208. *Construction of Elementary Solutions for I-hyperbolic Operators and Solutions with Small Singularities*

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In this note we treat the following problems:

(I) Construction of the elementary solution of the Cauchy problem for a hyperbolic differential operator (Theorem 3).

(II) The condition for I-hyperbolicity (Theorem 5).

(III) Construction of a solution of a (homogeneous) differential equation whose singular support on S^*M is contained in a bicharacteristic strip (Theorem 6). Here S^*M denotes the co-sphere or co-tangential sphere bundle of the underlying real analytic manifold M, which we take to be a domain in $\boldsymbol{R}^{n+1}$ containing the origin.

This paper is a summary of a part of forthcoming paper Kawai [7] in which details will be given. Throughout this note P will denote a linear partial differential operator of order m and of simple characteristics with analytic coefficients, whose principal symbol we denote by P_m.

We first state a theorem essentially due to Hamada [1], which generalizes the Cauchy-Kovalevsky theorem.

Theorem 1. *Let P be a partial differential operator with holomorphic coefficients defined near the origin of $\boldsymbol{C}^{n+1}$. (Hereafter we denote a point in C^{n+1} by $(t,z)=(t,z_1,\cdots,z_n)$ and assume $P_m(t,z;1,0)\neq 0$ near the origin.) We assume that the solutions $\tau=\tau_j(t,z;\xi)$ $(j=1,\cdots,m)$ of the equation $P_m(t,z;\tau,\xi)=0$ are mutually disjoint near $(t,z;\xi)=(0,0;\xi_0)$ and consider the following singular Cauchy problem:*

$$\text{(SC)}\qquad \begin{cases} P(t,z,\partial/\partial t,\partial/\partial z)u_k(t,z,y;\xi)=0\\ (\partial/\partial t)^j u_k(t,z,y;\xi)|_{t=0}=\delta_{jk}(1/<z-y,\xi>)^n \end{cases}$$
$$(0\leqq j,k\leqq m-1,|\xi|=|\xi_0|=1,|\xi-\xi_0|\ll 1).$$

Then (SC) *admits a unique local solution $u_k(t,z,y;\xi)$ which is a multivalued analytic function of $(t,z)\in\boldsymbol{C}^{n+1}$ defined outside $K^{(1)}(y,\xi)\cup\cdots\cup K^{(m)}(y,\xi)$; here $K^{(l)}(y,\xi)=\{(t,z)\,|\,\varphi^{(l)}(t,z,y;\xi)=0\}$, $l=1,\cdots,m$, denote the m (non-singular) characteristic surfaces of P_m passing through the intersection of complex hypersurfaces $t=0$ and $\langle z,\xi\rangle=\langle y,\xi\rangle$ in $\boldsymbol{C}^{n+1}$. $\varphi^{(l)}(t,z,y;\xi)$ denotes the corresponding characteristic function or the phase function satisfying $P_m(t,z;\operatorname{grad}_{(t,z)}\varphi^{(l)}(t,z,y;\xi))\equiv 0$. Furthermore u has the form $\sum_{l=1}^m u^{(l)}$, where the summand $u^{(l)}$*

is multivalued analytic except on a single $K^{(l)}$ *and satisfies the same equation* $Pu^{(l)}=0$. *The domain of the existence of* $u^{(l)}(t,z,y;\xi)$ *can be taken uniform with respect to* z *provided that* $|y|$ *is sufficiently small.*

We have further the following

Theorem 2. *Let* $1\leqq p\leqq m$ *and consider the following singular Cauchy problem*:

$$(\mathrm{SC})_p \qquad \begin{cases} P(t,z,\partial/\partial t,\partial/\partial z)u_k(t,z,y;\xi)=0 \\ (\partial/\partial t)^j u_k(t,z,y;\xi)|_{t=0}=\delta_{jk}(1/\langle z-y,\xi\rangle)^n \\ \qquad (0\leqq j,k\leqq p-1). \end{cases}$$

Then we can construct in a canonical way the solution of $(\mathrm{SC})_p$ *which is multivalued analytic outside* $K^{(1)}(y,\xi)\cup\cdots\cup K^{(p)}(y,\xi)$.

These two theorems are proved in an analogous way to Hamada [1], so we omit the details.

We treated the problem of hyperbolicity for convolution operators in Kawai [3], [4], [5] and constructed the elementary solution of Cauchy problem for hyperbolic differential operators with constant coefficients in Kawai [4], [6]. In this note we treat the same problem for the operators with analytic coefficients under the assumption of strict hyperbolicity.

It is easy to extend the following result to those operators satisfying the Levi condition (cf. Mizohata and Ohya [8]) but here we restrict ourselves to the simple characteristic case for the sake of simplicity.)

Theorem 3. *Let* P *be a strictly hyperbolic operator. Then we can construct the local elementary solutions for the Cauchy problem,* $E_k(t,x,y)$ $(k=0,\cdots,m-1)$, *which depend real analytically on* y *and satisfy the following*:

$$\begin{cases} P(t,x,\partial/\partial t,\partial/\partial x)E_k(t,x,y)=0 \\ (\partial/\partial t)^j E_k(t,x,y)|_{t=0}=\delta_{jk}\delta(x-y) \quad (j=0,\cdots,m-1). \end{cases}$$

Sketch of the proof. We use Theorem 1 to obtain $E_k(t,z,y;\xi)$ which satisfies (SC) with ξ in S^{n-1}, the cotangential sphere at y, and decompose E_k to $\sum_{l=1}^m E_k^{(l)}$ as remarked in Theorem 1. Then we can define a hyperfunction $E_k^{(l)}(t,x,y;\xi)$ by taking the boundary value of $E_k^{(l)}(t,z,y;\xi)$ from the complex domain $\{\mathrm{Im}\,\varphi^{(l)}(t,z,y;\xi)>0\}$. This is possible since $\varphi^{(l)}=0$ can be taken as a real equation for real y and real $\xi(\neq 0)$ by the assumption of hyperbolicity. Moreover S. S. $E_k^{(l)}(t_0,x,y_0;\xi_0)$ is contained in $\{(x,\eta)\,|\,\varphi^{(l)}(t_0,x,y_0;\xi_0)=0$ and $\eta=c\,\mathrm{grad}_x\varphi^{(l)}(t_0,x,y_0,\xi_0)$ $(c>0)\}$; here S. S. means the singular support on S^*M. (As to the notion of the singular support on S^*M, we refer the reader to Sato [9]-[13]. Now the local elementary solution stated in the theorem is given by

$$E_k(t,x,y)=\frac{(n-1)!}{(-2\pi i)^n}\sum_{l=1}^m\int E_k^{(l)}(t,x,y;\xi)\omega(\xi),$$

where $\omega(\xi)=\sum_{j=1}^{n}(-1)^{j-1}\xi_j d\xi_1\wedge\cdots\wedge d\xi_{j-1}\wedge d\xi_{j+1}\wedge\cdots\wedge d\xi_n$
is the volume element of S^{n-1}.

Remark. We can also show that the singular support of $E_k(t, x, y)$ is contained in the characteristic conoid issuing from y by using the theory of integration on the sheaf $\mathcal{C}$ (Sato [11]-[13]) and the properties of the characteristic function $\varphi^{(l)}(t, x, y; \xi)$. Combining a precise version of Holmgren's uniqueness theorem (Kawai [4], [5]) with this regularity property we conclude $E_k(t, x, y)$ has the trivial lacuna. A little weaker form of Holmgren's uniqueness theorem for hyperfunction solutions is also obtained by Schapira [14].

Theorem 4. *Assume that the following Cauchy problem* (C) *is solvable near the origin*:

(C) $\begin{cases} P(t, x, \partial/\partial t, \partial/\partial x)u(t, x)=0 \\ (\partial/\partial t)^j u(t, x)|_{t=0}=\mu_j(x), (0\leqq j\leqq m-1), \text{ where } \mu_j(x) \text{ are hyperfunctions arbitrarily given on the non-characteristic) initial surface} \\ \{t=0\}. \end{cases}$

Then $P_m(0, x; \tau, \xi)=0$ *has, as an equation for* τ, *at least one real solution for any* x *near the origin and any* ξ *in* S^{n-1}.

This theorem is an easy corollary of Sato's fundamental theorem concerning the regularity of the solutions. About Sato's fundamental theorem we refer the reader to Sato [9], [10] and for the proof of it to Sato [11], [12]. More refined version using the pseudo-differential operators will be given in forthcoming papers of Kashiwara and Kawai.

Theorem 5 (*I-hyperbolicity*). *Let* M_0 *denote the initial hypersurface or the domain in* $\{t=0\}=\{0\}\times \boldsymbol{R}^n\subset \boldsymbol{R}^{n+1}$, *and let* I *be an open set in* S^*M_0. *Assume that* $P_m(t, x; \tau, \xi)$ *has real coefficients and that the solutions* $\tau_j(0, x; \xi)$ $(j=1, \cdots, m)$ *of* $P_m(0, x; \tau, \xi)=0$ *are all real and distinct* whenever $(x, \xi)\in I$. *Then the following Cauchy problem* (C_I) *is locally solvable.*

(C_I) $\begin{cases} P(t, x; \partial/\partial t, \partial/\partial x)u(t, x)=0 \\ (\partial/\partial t)^j u(t, x)|_{t=0}=\mu_j(x), (j=0, \cdots, m-1), \text{ where } \mu_j(x) \text{ are hyperfunctions on } M_0 \text{ satisfying S.S. } \mu_j\ll I. \end{cases}$

The proof of this theorem is analogous to that of Theorem 3.

Example. Let $P=(\partial/\partial t)^2-(\partial/\partial x_1)^2+(\partial/\partial x_2)^2$ and $I=\{(x_1, x_2; \xi_1, \xi_2)| \xi_1>|\xi_2|\}$. Then P is I-hyperbolic.

Remark. This theorem connects the Cauchy-Kovalevsky thorem and the theory of hyperbolic operators. It seems that Voltera [15] treats the above example.

Theorem 6 (*Construction of a singular solution whose singular support is very small*). *Let* P_m *be of real coefficients, and denote by*

b one of its (real) bicharacteristic strip, which we regard as a curve in S^*M. Then there exists a hyperfunction u which satisfies $Pu=0$ and has non-void S.S. *u contained in b.*

Sketch of the proof. We choose the coordinate (t, x) so that the solution $\tau(t, x; \xi)$ of $P_m(t, x; \tau, \xi)=0$ is non-singular near $(0, 0; \xi_0)$, where $(0, 0, \tau(0, 0; \xi_0), \xi_0)$ is the initial data of the bicharacterisitic strip b. Then we can find $E(t, x, y; \xi)$ for which $PE=0$ and $E(0, x, y; \xi) =1/(\langle x-y, \xi\rangle+i0)^n$ hold, by employing Theorem 2 in an analogous way to Theorem 1 employed in the proof of Theorem 3. Taking $\mu(x)=1/\{\langle x, \xi_0\rangle+i(|x|^2-\langle x, \xi_0\rangle^2)+i0\}(|\xi_0|=1)$, we define $v(x) =\iint_{(y,\xi)\in I} E(t, x, y; \xi)\mu(y)dy\omega(\xi)$, where I is a small neighbourhood of $(0, \xi_0)$. From the way of construction of $E(t, x, y; \xi)$ together with the theory of integration on the sheaf $\mathcal{C}$ one concludes that S.S. $v\neq\phi$, S.S. $v\subset b$ and $Pv=\varphi$ with real analytic φ. Theorem 6 now follows immediately by replacing v by $u=v-v_0$, where v_0 denotes an analytic solution of $Pv_0=\varphi$ whose existence is obvious by the Cauchy-Kovalevsky theorem.

Remark. This theorem asserts that $u(x)$ is *real analytic* except on the single bicharacteristic curve. This improves a theorem of Hörmander [2] (Theorem 3.6) and Zerner [16]. We shall prove in Kawai [7] that this result is the best possible of the sort by constructing an elementary solution with good properties.

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