15. A Uniqueness Theorem for Symmetric Hyperbolic Systems of First Order in One Space Variables

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1. In [4], Segal has shown that a solution of the relativistic wave equation $u_{tt}-u_{xx}+\alpha u=0$, $\alpha>0$, which vanishes on the forward light rays $x^2-t^2=0$, $t>0$, vanishes identically under certain boundedness condition. This is quite different from the ordinary wave equation where one has a wide class of outgoing waves which vanish in a light cone. This result has been extended by Goodman [1] and Morawetz [3] to the equation $u_{tt}-\Delta u+\alpha u=0$ in three space variables. The condition on the solution is that energy integral $\int_{R^3}(|\nabla u|^2+|u_t|^2+\alpha|u|^2)dv$ is bounded. Recently, Taniguchi [5] has given another proof of the above result in one space variable under the stronger assumption of initial values and also proved the similar result for some first order symmetric hyperbolic systems in one space variable. This paper is intended to extend the result for hyperbolic systems in [5].

Let us consider the uniqueness of the Cauchy problems for hyperbolic systems of first order:

$$\text{(1.1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t}=A\dfrac{\partial u}{\partial x}+iBu \\ u(0,x)=u_0(x) \end{cases}$$

in a half space $\{(t,x)\,|\,t\geqq 0, x\in R^1\}$ where A and B are $N\times N$-constant symmetric matrices, and u is an N vector $(N\geqq 2)$.

We assume the following condition throughout this paper:

(I) $\det(\lambda I-(A\xi+B\eta))$ has real distinct zeros for any real $(\xi,\eta)\neq(0,0)$.

Then we obtain

Theorem. *Let u be a solution of* (1.1) *for $u_0(x)\in\mathcal{D}^1_{L^2}$. If $u(t,x)=0$ on $x+a_1t=0$ and $x+a_Nt=0$, where a_1 and a_N are the maximum and the minimum eigenvalues of matrix A, then $u(t,x)\equiv 0$.*

In [5], we assumed the condition: $\det(\lambda I-(A\xi+B\eta))$ has a form $\prod_{i=1}^{s}[\lambda^2-d_i^2(\xi^2+\eta^2)]$ for any real $(\xi,\eta)\neq(0,0)$, where d_i are positive and distinct, and $N=2S$. In this sense, this paper is an extension of the result in [5].

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2. We begin by defining the total energy and the energy inside an interval $[-a_1t-r, -a_Nt+r]$ at t:

$$E(t)=\int_{-\infty}^{\infty}|u(t,x)|^2dx \tag{2.1}$$

$$E(t:r)=\int_{-a_1t-r}^{-a_Nt+r}|u(t,x)|^2dx. \tag{2.2}$$

Here, $|u(t,x)|^2\equiv\langle u(t,x),u(t,x)\rangle=\sum_{i=1}^{N}u_i(t,x)\overline{u_i(t,x)}$ and r is a non-negative constant.

Lemma 1. *We assume the condition of the Theorem. Then we have, for $t_1\geqq t_2\geqq 0$ and $t\geqq 0$,*

$$\begin{cases}\text{(i)} & E(t)=E(0)\\ \text{(ii)} & E(t_1:r)\geqq E(t_2:r)\\ \text{(iii)} & E(t:0)=0.\end{cases} \tag{2.3}$$

Proof. Since $u_0(x)\in\mathcal{D}^1_{L^2}$, $u(t,x)$ is a solution of (1.1) such that $u(t,x)\in\mathcal{E}^0_t(\mathcal{D}^1_{L^2})\cap\mathcal{E}^1_t(L^2)$, (see, for example [2]).

$$\frac{d}{dt}E(t)=0$$

$$\frac{d}{dt}E(t:r)=\langle(A-a_NI)u(t,-a_Nt+r),u(t,-a_Nt+r)\rangle$$
$$+\langle(a_1I-A)u(t,-a_1t-r),u(t,-a_1t-r)\rangle\geqq 0.$$

Thus we obtain (i) and (ii). Finally, by the conditions $u(t,x)=0$ on $x+a_1t=0$ and $x+a_Nt=0$,

$$\frac{d}{dt}E(t:0)=0.$$

Therefore, we have the above conclusions. Q.E.D.

Lemma 2. *Let $p(\xi)$ be a real valued function in $C^1(R^1)$, and let $p'(\xi)$ have only finite zero points in any finite closed interval. Then, for any function $f(\xi)\in L^1(R^1)$,*

$$\int_{-\infty}^{\infty}e^{ip(\xi)t}f(\xi)d\xi\to 0\quad(t\to+\infty). \tag{2.4}$$

Proof. For any $\varepsilon>0$, there is a positive number l such that

$$\int_{-\infty}^{-l}|e^{ip(\xi)t}f(\xi)|d\xi<\frac{\varepsilon}{4},\ \int_{l}^{\infty}|e^{ip(\xi)t}f(\xi)|d\xi<\frac{\varepsilon}{4}. \tag{2.5}$$

From the assumptions, $p'(\xi)$ has only finite zero points on $[-l,l]$. We denote them by $\theta_1,\cdots,\theta_m$ and assume $(-l)<\theta_1<\cdots<\theta_m<l$ without loss of generality. For the ε fixed above, there is a $\delta>0$ such that

$$\delta<\frac{d}{2},\ d=\min\{|\theta_1-(-l)|,|\theta_2-\theta_1|,\cdots,|\theta_m-\theta_{m-1}|,|l-\theta_m|\}$$

and

$$\int_{\theta_i-\delta}^{\theta_i+\delta}|e^{ip(\xi)t}f(\xi)|d\xi<\frac{\varepsilon}{8(m+1)}\quad(i=1,\cdots,m). \tag{2.6}$$

Then we have only to estimate $(m+1)$ formulas such as

$$\int_h^k e^{ip(\xi)t} f(\xi) d\xi \qquad \text{for} \quad t \to +\infty$$

where h and k are real numbers, and $p'(\xi) \neq 0$ on $[h-\delta/2, k+\delta/2]$. Since $g(\xi) = f(\xi)/p'(\xi) \in L^1(h-\delta/2, k+\delta/2)$, there is a function $h(\xi) \in C^1(h-\delta/4, k+\delta/4)$ such that

$$\text{(2.7)} \quad \int_h^k |g(\xi) - h(\xi)| d\xi < \frac{\varepsilon}{16(m+1)(M+1)}, \quad M = \max_{h \le \xi \le k} |p'(\xi)|.$$

$$\therefore \quad \int_h^k e^{ip(\xi)t} f(\xi) d\xi = \int_h^k e^{ip(\xi)t} p'(\xi) \left[\frac{f(\xi)}{p'(\xi)} - h(\xi) \right] d\xi$$
$$+ \left[\frac{(-1)}{it} e^{ip(\xi)t} h(\xi) \right]_h^k + \frac{1}{it} \int_h^k e^{ip(\xi)t} h'(\xi) d\xi.$$

So, by the similar method to the proof of Riemann-Lebesgue's Theorem, we can show that $^\exists t_0 > 0$,

$$\left| \int_h^k e^{ip(\xi)t} f(\xi) d\xi \right| < \frac{\varepsilon}{8(m+1)}, \quad ^\forall t > t_0.$$

$$\therefore \quad \left| \int_{-\infty}^{\infty} e^{ip(\xi)t} f(\xi) d\xi \right| < \varepsilon, \quad ^\forall t > t_0.$$

Here ε is any positive number. Therefore Lemma 2 is proved. Q.E.D.

Lemma 3. *Under the condition* (I), *the characteristic equation*

$$\text{(2.8)} \qquad \det(\lambda I - (A\xi + B)) = 0, \quad ^\forall \xi \in R^1$$

has no roots of the form of $(a_1\xi + c)$ *or* $(a_N\xi + c)$, *where* c *is a real constant.*

Proof. As the proof is similar, we shall prove that (2.8) has no roots of the form of $(a_1\xi + c)$. We assume that $(a_1\xi + c)$ is a root of (2.8). Any root $\lambda_j(\xi, \eta)$ $(1 \leqq j \leqq N)$ of $\det(\lambda I - (A\xi + B\eta)) = 0$ for any real $(\xi, \eta) \neq (0, 0)$ has the properties $\lambda_j(\xi, \eta) = \sqrt{\xi^2 + \eta^2}\, \lambda_j\left(\frac{\xi}{\sqrt{\xi^2 + \eta^2}}, \frac{\eta}{\sqrt{\xi^2 + \eta^2}} \right)$ and $\lambda_j(-\xi, -\eta) = -\lambda_{N-j+1}(\xi, \eta)$, $(\lambda_1 > \lambda_2 > \cdots > \lambda_N)$. Let $\lambda(\xi, 1)$ be $(a_1\xi + c)$. Then, by the above properties, $\lambda(1, 0) = a_1 = \lambda_1(1, 0)$ and $\lambda(-1, 0) = -a_1 = \lambda_N(-1, 0)$. This is a contradiction. Therefore Lemma 3 holds. Q.E.D.

We remark that under Condition (I) any root of the characteristic equation (2.8) is extended to a function which is regular in a domain $(-L, L) \times (-iK(L), iK(L))$ in C^1 where L is any positive number and $K(L)$ is a positive number dependent on L. This fact is used to show that $(-a_1\xi + \lambda_j(\xi))$ in (3.8) satisfies the condition of Lemma 2.

3. Proof of Theorem. Assume the contrary. We choose a sufficiently large positive number r such as

$$\text{(3.1)} \qquad E(0) - E(0:r) < \frac{1}{3} E(0).$$

If we can show the inequality

$$E(t:r)<\frac{1}{3}E(0) \tag{3.2}$$

for sufficiently large positive number t, then by (2.1)

$$E(0)\leqq E(0)-E(0:r)+E(t:r)<\frac{2}{3}E(0), \tag{3.3}$$

and we obtain a contradiction. So we shall prove the inequality (3.2). By Lemma 1,

$$E(t:r)=\int_{-a_1t-r}^{-a_1t}|u|^2dx+\int_{-a_Nt}^{-a_Nt+r}|u|^2dx. \tag{3.4}$$

The integral interval of (3.4) is $2r$ independent of t. To obtain (3.2), we have only to prove that $u(t,-a_1t-\alpha)$ and $u(t,-a_Nt+\beta)\to 0$ uniformly in α and β $(0\leqq\alpha,\beta\leqq r)$ as $t\to+\infty$. As the proof is similar, we shall prove that $u(t,-a_1t-\alpha)\to 0$ uniformly in α as $t\to+\infty$.

Using the inverse Fourier transform, a solution of (1.1) at $x=(-a_1t-\alpha)$ is represented by

$$u(t,-a_1t-\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\xi(-a_1t-\alpha)}e^{i(A\xi+B)t}\hat{u}_0(\xi)d\xi \tag{3.5}$$

where $\hat{u}_0(\xi)$ is the Fourier transform of $u_0(x)$. Let $C(\xi)$ be $(A\xi+B)$. By Condition (I) there is a unitary matrix $U(\xi)$ such that $U(\xi)$ is in $C^\infty(R^1)\cap\mathscr{B}^0(R^1)$ and $U(\xi)C(\xi)U^*(\xi)$ is a diagonal matrix $D(\xi)$, [2]. We write that

$$D(\xi)=\begin{pmatrix}\lambda_1(\xi) & & 0\\ & \ddots & \\ 0 & & \lambda_N(\xi)\end{pmatrix}. \tag{3.6}$$

Therefore

$$\begin{aligned} w(t,\xi)&\equiv e^{i\xi(-a_1t-\alpha)}e^{i(A\xi+B)t}\hat{u}_0(\xi)\\ &=U^*(\xi)\begin{pmatrix}e^{i(-a_1\xi+\lambda_1(\xi))t} & & 0\\ & \ddots & \\ 0 & & e^{i(-a_1\xi+\lambda_N(\xi))t}\end{pmatrix}U(\xi)e^{-i\alpha\xi}\hat{u}_0(\xi). \end{aligned} \tag{3.7}$$

So, any element of a vector $w(t,\xi)$ is finite linear combination of functions such that

$$e^{i(-a_1\xi+\lambda_j(\xi))t}e^{-i\alpha\xi}v(\xi),\quad v(\xi)\in L^1(R^1)$$

where j varies 1 to N and $0\leqq\alpha\leqq r$. Therefore, to obtain (3.2), we have only to prove that the formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(-a_1\xi+\lambda_j(\xi))t}e^{-i\alpha\xi}v(\xi)d\xi,\quad v(\xi)\in L^1(R^1) \tag{3.8}$$

$\to 0$ uniformly in α as $t\to+\infty$. Since $u_0(x)\in\mathscr{D}^1_{L^2}$, $\hat{u}_0(\xi)\in L^1(R^1)$. Using Lemma 2 and Lemma 3, there is a $t_1>0$ such that the inequality (3.2) holds for any $t>t_1$. Then we have (3.3). This is a contradiction. Therefore the theorem holds. Q.E.D.

Remark. Instead of Condition (I), we assume the condition:

(II) $\det(\lambda I-(A\xi+B))$ has real and distinct zeros for any real ξ. Then, we can not generally obtain the theorem by our method because Lemma 3 does not necessarily hold. We show such an example.

$$\frac{\partial u}{\partial t}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\frac{\partial u}{\partial x}+i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}u.$$

In fact,
$$\lambda(\xi)=\xi\pm 1.$$

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