45. *On the Spectrum of the Laplace-Beltrami Operator on a Non-Compact Surface*

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1. Introduction and preliminaries. In this note we consider the Laplace-Beltrami operator Δ on a non-compact surface M in the Euclidean 3-space E^3. Our purpose is to show that, under certain assumptions on M, Δ has no eigenvalue as an operator in the Hilbert space $L^2(M)$. Several authors have worked on the eigenvalue problems for Δ or the Schrödinger operators $-\Delta+q$ in E^n or in certain unbounded subdomains of E^n. Our problem differs from theirs in that it cannot necessarily be reduced to the problem in the flat Euclidean space. However, suggestions of our method can be found in their works, especially Rellich [1] (see also Eidus [2] p. 42, Theorem 10).

Let M be a surface of class C^2 in E^3. Let (g_{ij}) be the Riemann metric tensor on M, $(g^{ij})=(g_{ij})^{-1}$ and $G=det(g_{ij})$. The Laplace-Beltrami operator Δ on M is given by

$$(1)\qquad \Delta u=\sum_{i,j=1}^{2}\frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}\left(\sqrt{G}\,g^{ij}\frac{\partial u}{\partial x^j}\right)\qquad (\text{for } u\in C^2(M))$$

in any local coordinate system (x^1, x^2). $L^2(M)$ is the totality of measurable functions which are square integrable on M, that is, $L^2(M)$ is a Hilbert space with the scalar product

$$(\phi,\psi)=\int\phi\bar{\psi}dM,$$

where dM is the surface element of M: $dM=\sqrt{G}dx^1dx^2$. D_1 is the completion of $C_0^2(M)$ (C^2 functions on M with compact supports) with regard to the norm

$$\|f\|_1^2=\|f\|^2+\int|\text{grad}\, f|^2\,dM,$$

where $\|f\|^2=(f,f)$ and $|\text{grad}\, f|^2=\sum_{i,j=1}^2 g^{ij}\partial f/\partial x^i\cdot\overline{\partial f/\partial x^j}$. D_1' is the dual space of D_1 and $L^2(M)$ is imbedded in D_1' in the usual way. We define Δu for $u\in D_1$ as follows; $F\in D_1'$ equals to Δu if

$$(2)\qquad F(\phi)=\int u\Delta\phi dM$$

for any $\phi\in C_0^2(M)$. Let L be the operator with the domain $D(L)=\{f: f\in D_1, \Delta f\in L^2(M)\}$ and $Lu=\Delta u$.

Lemma 1. *L is a non-positive definite self-adjoint operator in $L^2(M)$.*

Proof (cf. [3] p. 137). $((I-\Delta)\phi,\phi)\geq(\phi,\phi)$ for any $\phi\in C_0^2(M)$. In view of (2), this inequality holds for any $\phi\in D_1(M)$. By Riesz' theorem, $I-\Delta$ maps D_1 one-to-one onto D_1'. We can see $(I-L)^{-1}=(I-\Delta)^{-1}|_{L^2(M)}$ and this is a bounded symmetric operator in $L^2(M)$. This proves L is self-adjoint. q.e.d.

Now we assume that M is a surface of revolution in E^3 represented by

$$x=r(s)\cos\theta,\ y=r(s)\sin\theta,\ z=h(s),\ r(0)=h(0)=0,\qquad (0\leq s<\infty\,;\,0\leq\theta<2\pi),$$

where the parameter s is the length from $(0,0,0)$ to (x,y,z) along the generating line of M. The Riemann metric induced on M from E^3 is given by $g_{11}=1$, $g_{22}=r^2$ and $g_{12}=g_{21}=0$, so we have

$$\Delta u=\frac{1}{r}\frac{\partial}{\partial s}\left(r\frac{\partial u}{\partial s}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\qquad (u\in C^2(M)).\tag{3}$$

Now we shall show

Lemma 2. *Let M_t be the subdomain of M characterized by $s\in[0,t)$, and Γ_t the boundary of M_t. If w is a smooth function on M and $\Delta w=\lambda w$ for a real number λ, then*

$$2\lambda\int_{M_t}\frac{dr}{ds}|w|^2\,dM=r(t)\int_{\Gamma_t}\left(\lambda|w|^2-\left|\frac{\partial w}{\partial s}\right|^2+\frac{1}{r^2}\left|\frac{\partial w}{\partial\theta}\right|^2\right)d\Gamma\tag{4}$$

$$(dM=rdsd\theta,\ d\Gamma=rd\theta).$$

Proof. Put

$$I=\int_{M_t}r\left(\frac{\partial w}{\partial s}\Delta\bar{w}+\frac{\partial\bar{w}}{\partial s}\Delta w\right)dM.$$

Using (3) and integrating by parts, we have

$$I=\int_{\Gamma_t}r\left(\left|\frac{\partial w}{\partial s}\right|^2-\frac{1}{r^2}\left|\frac{\partial w}{\partial\theta}\right|^2\right)d\Gamma.\tag{5}$$

On the other hand, in view of $\Delta w=\lambda w$, we see that

$$I=\lambda\int_{M_t}r\left(\frac{\partial w}{\partial s}\bar{w}+\frac{\partial\bar{w}}{\partial s}w\right)dM=\lambda\int_{M_t}r\frac{\partial}{\partial s}|w|^2\,dM$$

$$=-2\lambda\int_{M_t}\frac{dr}{ds}|w|^2\,dM+\lambda\int_{\Gamma_t}r|w|^2\,d\Gamma.$$

This, together with (5), proves the lemma.

2. Non-existence of eigenvalues.

Theorem. *If $dr/ds\geq 0$ for $0\leq s<\infty$, then L has no eigenvalues.*

Proof. Let $Lu=\lambda u$, $u\in D(L)$ and λ be real. Note that the ellipticity of Δ guarantees the smoothness of u. Since L is non-positive definite by Lemma 1, the eigenvalue λ must be non-positive.

First let us assume $\lambda=0$. Then integrating by parts gives

$$0=\int_{\Gamma_t}u\Delta\bar{u}dM=-\int_{M_t}|\operatorname{grad}u|^2\,dM+\int_{\Gamma_t}u\frac{\partial\bar{u}}{\partial s}d\Gamma.$$

Here we can choose a sequence $t_n\to\infty$ such that the second term of the right side goes to zero if we take $t=t_n$ in the above equation, since $u\partial\bar{u}/\partial s$ is summable over M. Thus we have

$$0=\int_M |\operatorname{grad} u|^2 dM,$$

whence follows $\operatorname{grad} u\equiv 0$, and so $u\equiv$const. Since u must be in $L^2(M)$, $u\equiv 0$ which shows that $\lambda=0$ cannot be an eigenvalue of L.

Next let us assume $\lambda<0$. By Lemma 2 with $w=u$, we have

$$(6)\qquad 2\lambda\int_{M_t}\frac{dr}{ds}|u|^2 dM=r(t)\int_{\Gamma_t}\left(\lambda|u|^2-\left|\frac{\partial u}{\partial s}\right|^2+\frac{1}{r^2}\left|\frac{\partial u}{\partial\theta}\right|^2\right)d\Gamma.$$

Since $u\in D_1$, $\lambda|u|^2-|\partial u/\partial s|^2+(1/r^2)|\partial u/\partial\theta|^2$ is summable over M, which implies that we can choose a sequence $\{t_n\}$ tending to ∞ such that

$$t_n\int_{\Gamma_{t_n}}\left(\lambda|u|^2-\left|\frac{\partial u}{\partial s}\right|^2+\frac{1}{r^2}\left|\frac{\partial u}{\partial\theta}\right|^2\right)d\Gamma$$

tends to zero as $n\to\infty$. Noting that $r(t_n)\leq t_n$, therefore, if we let t tend to ∞ through this sequence $\{t_n\}$ in (6), then we have

$$\lambda\int_M\frac{dr}{ds}|u|^2 dM=0.$$

This gives $u\equiv 0$ on the subset of M in which $dr/ds>0$. The subset is non-empty and open, because $r(0)=0$ and $r(s)>0$ for $s\neq 0$. The unique continuation theorem for second order elliptic equations ([4] or [3] pp. 230 and 339) enables us to conclude $u\equiv 0$ on the whole of M.

References

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