60. *An Extension of an Integral. I*

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1. **Introduction.** An integral σ with respect to an integral structure Γ was defined in the author [3]. An example of integrals of this type is 1-dimensional (or generally n-dimensional) Lebesgue integral of bounded measurable functions over measure-finite measurable sets (see Introduction in [1]). In this case, however, we can not deal with such integrals as

$$\int_{-\infty}^{\infty} f(x)dx \qquad \text{where } f(x)=\begin{cases} x^{-2} & (1<x) \\ x^{-1/2} & (0<x\leqq 1) \\ 0 & (x\leqq 0) \end{cases}$$

in our way. We shall extend in this paper the integral σ to an 'integral' $\bar{\sigma}$ and then integrals of the above type may be dealt in terms of $\bar{\sigma}$.

2. **Extension theorems.** *Let $\Gamma=(\Lambda;\ \mathcal{S},\mathcal{G},Q)$ be an integral structure and σ an integral with respect to Γ.*

Denote by $\mathcal{M},\mathcal{F}$, and J the total ring, the total functional group, and the third group, respectively, of Λ and let $\bar{\mathcal{S}}$ be the σ-ring generated by $\mathcal{S}$.

Let Ω be the set of all elements (X, f, μ) of $\mathcal{M}\times\mathcal{F}\times Q$ satisfying the following conditions:

1) There exist $X_i \in \mathcal{S}, i=1,2,\cdots$, such that $X_i f \in \mathcal{G}$ for any i and such that $X_i \uparrow X$ $(i\to\infty)$.

2) If $X_i^{(k)} \in \mathcal{S}, X_i^{(k)} f \in \mathcal{G}$, for $i=1,2,\cdots$, and if $X_i^{(k)}\uparrow X(i\to\infty)$, where $k=1,2$, then for any neighborhood V of $0 \in J$ there exists a positive integer n such that $\sigma(X_l^{(1)}, X_l^{(1)}uf, \mu)-\sigma(X_m^{(2)}, X_m^{(2)}f, \mu) \in V$ for any $l\geqq n$ and $m \geqq n$.

The set Ω defined above will be called the *carrier* of Γ.

Let us assume the following:

1) $\sigma(X_i, g, \mu)\to 0$ $(i\to\infty)$ *for* $X_i \in \mathcal{S}, i=1,2,\cdots$, *such that* $X_i \downarrow 0$ $(i\to\infty)$, *for any* $g\in\mathcal{G}$ *and* $\mu\in Q$.

2) $\mathcal{S}$ *is a pseudo-σ-ring.*

3) J *is Hausdorff and complete.*

Then we have the following theorems, which will be proved in Part II of this paper.

Theorem 1. *Under the above assumptions,*

1) $\mathcal{S}\times\mathcal{G}\times Q\subset\Omega\subset\bar{\mathcal{S}}\times\mathcal{F}\times Q$.

2) *For any $X, Y \in \bar{\mathcal{S}}, f\in\mathcal{F}$, and $\mu\in Q$, it holds that $(XY, f, \mu)\in\Omega$*

if and only if $(X, Yf, \mu) \in \Omega$.

3) *For any* $f \in \mathcal{F}$ *and* $\mu \in Q$, *the set* $\mathcal{S}(f, \mu)=\{X \mid (X, f, \mu) \in \Omega\}$ *is an ideal of* $\bar{\mathcal{S}}$ *and consequently is a pseudo-*σ*-ring.*

4) *For any* $X \in \bar{\mathcal{S}}$ *and* $\mu \in Q$, *the set* $\mathcal{G}(X, \mu)=\{f \mid (X, f, \mu) \in \Omega\}$ *is a subgroup of* $\mathcal{F}$.

Theorem 2. *There exists a unique map* $\bar{\sigma}$ *of* Ω *into* J *satisfying the conditions*:

1) $\bar{\sigma}$ *is an extension of* σ.

2) *For any* $X, Y \in \bar{\mathcal{S}}, f \in \mathcal{F}$, *and* $\mu \in Q$, $(XY, f, \mu) \in \Omega$ *implies* $\bar{\sigma}(XY, f, \mu)=\bar{\sigma}(X, Yf, \mu)$.

3) *For any* $f \in \mathcal{F}$ *and* $\mu \in Q$, *the map* $\bar{\sigma}_{f,\mu}(X)=\bar{\sigma}(X,f,\mu)$ *on* $\mathcal{S}(f,\mu)$ *is a measure.*

Further, this map $\bar{\sigma}$ *satisfies the following*:

4) *For any* $X \in \bar{\mathcal{S}}$ *and* $\mu \in Q$, *the map* $\bar{\sigma}_{X,\mu}(f)=\bar{\sigma}(X,f,\mu)$ *on* $\mathcal{G}(X,\mu)$ *is a homomorphism.*

The map $\bar{\sigma}$ in Theorem 2 will be called the *measure extension* of σ.

To show that the domain of $\bar{\sigma}$ is sufficiently large, in a sense, we shall prove the next proposition. Note that the uniqueness of $\bar{\sigma}$ in Theorem 2 is easily verified by means of (i), (ii), and (iv) in the proof of the proposition.

Proposition 1. *Consider the following conditions on a pair*(Ω', σ'):

1) $\Omega' \subset \mathcal{M} \times \mathcal{F} \times Q$ *and* σ' *is a map of* Ω' *into* J.

2) *For any* $(X, f, \mu) \in \Omega'$, *there exist* $X_i \in \mathcal{S}, i=1,2,\cdots$, *such that* $X_i f \in \mathcal{G}$ *for any* i *and such that* $X_i \uparrow X$ $(i \to \infty)$.

3) *For* $X \in \mathcal{S}$ *and* $f \in \mathcal{F}$ *such that* $Xf \in \mathcal{G}$, *and for any* $\mu \in Q$, *we have* (a) $(X, f, \mu) \in \Omega'$ *and* (b) $\sigma'(X, f, \mu)=\sigma(X, Xf, \mu)$.

4) *For any* $f \in \mathcal{F}$ *and* $\mu \in Q$, (a′) *the set* $\mathcal{S}'(f,\mu)=\{X \mid (X,f,\mu) \in \Omega'\}$ *is a subring of* $\mathcal{M}$ *and* (b′) *the map* $\sigma'_{f,\mu}(X)=\sigma'(X, f, \mu)$ *on* $\mathcal{S}'(f, \mu)$ *is a measure.*

Then a necessary and sufficient condition for a pair (Ω', σ') *to satisfy the above conditions is to be such a pair that* Ω' *is a subset of* Ω *satisfying* (a) *in* 3) *and* (a′) *in* 4) *and that* σ' *is the restriction of* $\bar{\sigma}$ *on* Ω'. *Further,* Ω *satisfies* (a) *and* (a′).

Proof. (i) The sufficiency is easily verified even if we assume that $\bar{\sigma}$ is an *arbitrary* map satisfying the conditions on $\bar{\sigma}$: 1), 2), and 3) in Theorem 2. To prove the necessity, let us show that (ii) if (Ω', σ') and (Ω', σ'') both satisfy the conditions, then $\sigma'=\sigma''$. For $(X, f, \mu) \in \Omega'$ and for $X_i \in \mathcal{S}$, $i=1,2,\cdots$, such that $X_i f \in \mathcal{G}$ for any i and such that $X_i \uparrow X$ $(i \to \infty)$, we have $\sigma'(X,f,\mu)=\sigma'_{f,\mu}(X)=\lim_{i\to\infty} \sigma'_{f,\mu}(X_i)=\lim_{i\to\infty} \sigma'(X_i,f,\mu)$ $=\lim_{i\to\infty} \sigma(X_i, X_i f, \mu)$, and this implies (ii). Next let us show that (iii) Ω' is a subset of Ω. Let (X, f, μ) be an element of Ω' and suppose that $X_i^{(k)} \in \mathcal{S}, X_i^{(k)} f \in \mathcal{G}$, for $i=1,2,\cdots$, and that $X_i^{(k)} \uparrow X$ $(i \to \infty)$, where $k=1$,

2. For given neighbourhood V of $0 \in J$, there exists a neighbourhood U of $0 \in J$ such that $U-U \subset V$. Since $\lim_{i\to\infty} \sigma(X_i^{(k)}, X_i^{(k)}f, \mu) = \sigma'(X, f, \mu)$, we have n_k such that $\sigma(X_i^{(k)}, X_i^{(k)}f, \mu) - \sigma'(X, f, \mu) \in U$ for any $i \geqq n_k$. For $n = \max(n_1, n_2)$ and for any $l \geqq n$ and $m \geqq n$, we have $\sigma(X_l^{(1)}, X_l^{(1)}f, \mu) - \sigma(X_m^{(2)}, X_m^{(2)}f, \mu) \in V$ and this implies (iii). Now let us show that σ' is the restriction of $\bar{\sigma}$. Let σ'' be the restriction of $\bar{\sigma}$ on Ω'. Then the pair (Ω', σ'') satisfies the conditions and hence (ii) implies that $\sigma' = \sigma''$. Finally let us show that (iv) Ω satisfies (a) and (a'). For $X_i \in \mathcal{S}, i=1, 2, \cdots$, such that $X_i \uparrow X$ $(i \to \infty)$, it follows that $\lim_{i\to\infty} \sigma(X_i, X_i f, \mu) = \lim_{i\to\infty} \sigma(X_i, Xf, \mu) = \sigma(X, Xf, \mu)$ and this implies that Ω satisfies (a). That Ω satisfies (a') follows from Theorem 1. Thus the proposition is proved.

3. Lemmas. In this section we shall give some lemmas to prove the theorems in section 2.

Assumption 1. *M is a set and $\mathcal{M}$ is the ring of all subsets of M. A subring $\mathcal{S}$ of $\mathcal{M}$ is a pseudo-σ-ring.*

Let Σ be the set of all maps ξ, defined on the set of all positive integers N and taking values in $\mathcal{S}$, such that $\xi(n) \subset \xi(n+1)$ for all $n \in N$. Put $\bar{\xi} = \bigcup_{n=1}^{\infty} \xi(n)$ for $\xi \in \Sigma$, and $\bar{\Theta} = \{\bar{\xi} \mid \xi \in \Sigma\}$ for $\Theta \subset \Sigma$. Then we have

Lemma 1. *$\bar{\Sigma} = \{\bigcup_{n=1}^{\infty} X_n \mid X_n \in \mathcal{S}, n=1, 2, \cdots\}$ and $\bar{\Sigma}$ is the sub-σ-ring of $\mathcal{M}$ generated by $\mathcal{S}$.*

Corollary. *$\mathcal{S}$ is an ideal of $\bar{\Sigma}$.*

For $X \in \bar{\Sigma}$ and for $\xi_i \in \Sigma, i=0, 1, \cdots, k$, let us define maps $X\xi_0$, $\xi_0\xi_1 \cdots \xi_k$ and $\xi_0 + \xi_1 + \cdots + \xi_k$ of N into $\mathcal{S}$ by

1) $(X\xi_0)(n) = X(\xi_0(n))$

2) $(\xi_0\xi_1 \cdots \xi_k)(n) = \xi_0(n)\xi_1(n) \cdots \xi_k(n)$

3) $(\xi_0 + \xi_1 + \cdots + \xi_k)(n) = \xi_0(n) + \xi_1(n) + \cdots + \xi_k(n)$

for any $n \in N$, respectively.

Lemma 2. *For $X \in \bar{\Sigma}$ and for $\xi_i \in \Sigma, i=0, 1, \cdots, k$, we have*

1) *$X\xi_0 \in \Sigma$ and $\overline{X\xi_0} = X\bar{\xi}_0$*

2) *$\xi_0\xi_1 \cdots \xi_k \in \Sigma$ and $\overline{\xi_0\xi_1 \cdots \xi_k} = \bar{\xi}_0\bar{\xi}_1 \cdots \bar{\xi}_k$*

3) *$\bar{\xi}_i\bar{\xi}_j = 0$ $(i \neq j)$ implies that $\xi_0 + \xi_1 + \cdots + \xi_k \in \Sigma$ and that $\overline{\xi_0 + \xi_1 + \cdots + \xi_k} = \bar{\xi}_0 + \bar{\xi}_1 + \cdots + \bar{\xi}_k$.*

Assumption 2. *$(\mathcal{S}, \mathcal{F}, J)$ is an abstract integral structure* [1] *and $\mathcal{G}$ is an $\mathcal{S}$-invariant subgroup of $\mathcal{F}$.*

Note that $(\mathcal{S}, \mathcal{G}, J)$ is also an abstract integral structure.

For each $f \in \mathcal{F}$, denote the sets $\{X \mid X \in \mathcal{S}, Xf \in \mathcal{G}\}$ and $\{\xi \mid \xi \in \Sigma, \xi(n)f \in \mathcal{G}$ for any $n \in N\}$ by $\mathcal{R}(f)$ and $\Sigma(f)$, respectively. We can write $\Sigma(f) = \{\xi \mid \xi \in \Sigma, \xi(n) \in \mathcal{R}(f)$ for any $n \in N\}$.

Lemma 3. *$\mathcal{R}(g) = \mathcal{S}$ and $\Sigma(g) = \Sigma$ for any $g \in \mathcal{G}$.*

Lemma 4. *For any $f \in \mathcal{F}$, $\mathcal{R}(f)$ is an ideal of $\bar{\Sigma}$ and is a pseudo-σ-ring.*

Proof. 1) It immediately follows that $0 \in \mathcal{R}(f) \subset \mathcal{S} \subset \bar{\Sigma}$. 2) For

$X \in \mathcal{R}(f)$ and for $Y \in \bar{\Sigma}$, Corollary to Lemma 1 implies that $XY \in \mathcal{S}$ and it holds that $(XY)f=(XY)(Xf) \in \mathcal{S}\mathcal{G} \subset \mathcal{G}$. Hence, $XY \in \mathcal{R}(f)$ for any $X \in \mathcal{R}(f)$ and $Y \in \bar{\Sigma}$. 3) For $X, Y \in \mathcal{R}(f)$ such that $XY=0$, we have $X+Y \in \mathcal{S}$, $(X+Y)f=Xf+Yf \in \mathcal{G}$, and thus we have $X+Y \in \mathcal{R}(f)$. 1), 2), and 3) above imply that $\mathcal{R}(f)$ is an ideal of $\bar{\Sigma}$. That $\mathcal{R}(f)$ is a pseudo-σ-ring follows from the fact that $\bar{\Sigma}$ is a σ-ring.

Corollary. *$\mathcal{R}(f)$ is an ideal of $\mathcal{S}$ for any $f \in \mathcal{F}$.*

Lemma 5. $\overline{\Sigma(f)}=\{\bigcup_{n=1}^{\infty} X_n \mid X_n \in \mathcal{R}(f), n=1,2,\cdots\}$ *for any $f \in \mathcal{F}$.*

Lemma 6. *For any $f \in \mathcal{F}$, $\overline{\Sigma(f)}$ is an ideal of $\bar{\Sigma}$ and is a σ-ring.*

Proof. This follows from 1), 2), and 3), below. 1) $0 \in \overline{\Sigma(f)} \subset \bar{\Sigma}$. 2) For $X \in \overline{\Sigma(f)}$ and $Y \in \bar{\Sigma}$, we have an element ξ of $\Sigma(f)$ such that $\bar{\xi}=X$. That $\mathcal{R}(f)$ is an ideal of $\bar{\Sigma}$ implies that $(Y\xi)(n)=Y(\xi(n)) \in \mathcal{R}(f)$ for each n and thus we have $Y\xi \in \Sigma(f)$. Hence $YX=Y\bar{\xi}=\overline{Y\xi} \in \overline{\Sigma(f)}$. 3) It holds that $\bigcup_{n=1}^{\infty} X_n \in \overline{\Sigma(f)}$ for $X_n \in \overline{\Sigma(f)}$, $n=1,2,\cdots$, which follows from Lemma 5.

Corollary 1. *$\mathcal{R}(f)$ is an ideal of $\overline{\Sigma(f)}$ for any $f \in \mathcal{F}$.*

Corollary 2. *If $f \in \mathcal{F}$, $X \in \bar{\Sigma}$, $\xi \in \Sigma(f)$, and if $\eta \in \Sigma$, we have*

1) $X\xi \in \Sigma(f)$
2) $\xi\eta \in \Sigma(f)$
3) $\bar{\xi}\bar{\eta}=0$ *and* $\eta \in \Sigma(f)$ *imply that* $\xi+\eta \in \Sigma(f)$.

Proof. For each n, we have 1) $(X\xi)(n)=X(\xi(n)) \in \bar{\Sigma}\mathcal{R}(f) \subset \mathcal{R}(f)$, 2) $(\xi\eta)(n)=\xi(n)\eta(n) \in \mathcal{R}(f)\mathcal{S} \subset \mathcal{R}(f)$, and 3) $(\xi+\eta)(n)=\xi(n)+\eta(n) \in \mathcal{R}(f)$.

Lemma 7. *$XY \in \mathcal{R}(f)$ if and only if $X \in \mathcal{R}(Yf)$, for any $f \in \mathcal{F}$, $X \in \mathcal{S}$ and $Y \in \bar{\Sigma}$.*

Corollary. *$X\xi \in \Sigma(f)$ if and only if $\xi \in \Sigma(Xf)$, for any $f \in \mathcal{F}, X \in \bar{\Sigma}$ and $\xi \in \Sigma$.*

Lemma 8. *If $f \in \mathcal{F}$, $\zeta \in \Sigma(f)$, and if $\bar{\zeta}=XY$ for $X, Y \in \bar{\Sigma}$, then we have an element ξ of $\Sigma(Yf)$ such that $\bar{\xi}=X$ and $\zeta=Y\xi$.*

Proof. Let η be an element of Σ such that $\bar{\eta}=X$. Put $\xi=\eta+Y\eta+\zeta$. Then we have $\xi(n) \in \mathcal{S}$, which follows from the fact that $\mathcal{S}$ is an ideal of $\bar{\Sigma}$, and we have $\xi(n)=(\eta(n)-Y) \cup \zeta(n) \subset \xi(n+1)$, for each n. This implies that $\xi \in \Sigma$. It follows that $(Y\xi)(n)=Y(\xi(n))=Y(\zeta(n))=\zeta(n)$ and this implies that $\zeta=Y\xi$. Since $\xi(n)(Yf)=((Y\xi)(n))f=\zeta(n)f \in \mathcal{G}$, we have $\xi \in \Sigma(Yf)$. Finally we have $\bar{\xi}=\bigcup_{n=1}^{\infty}\xi(n)=\bigcup_{n=1}^{\infty}((\eta(n)-Y)\cup\zeta(n)) =(\bigcup_{n=1}^{\infty}(\eta(n)-Y)) \cup (\bigcup_{n=1}^{\infty}\zeta(n)) = (\bigcup_{n=1}^{\infty}\eta(n)-Y) \cup \bar{\zeta} = (\bar{\eta}-Y) \cup XY =(X-Y) \cup XY=X$.

Now put $\tilde{\Omega}=\{(\bar{\xi}, f) \mid f \in \mathcal{F}, \xi \in \Sigma(f)\}$. Then we have

Lemma 9. $\overline{\Sigma(f)}=\{X \mid (X, f) \in \tilde{\Omega}\}$ *for any $f \in \mathcal{F}$.*

Lemma 10. *If $f_i \in \mathcal{F}$ and if $\xi_i \in \Sigma(f_i)$, $i=1,2,\cdots,k$, then $\xi_1\xi_2 \cdots \xi_k \in \bigcap_{i=1}^{k}\Sigma(f_i)$.*

Proof. This follows from Lemma 2 and Corollary 2 to Lemma 6.

Corollary 1. *If $f_i \in \mathcal{F}$, $\xi_i \in \Sigma(f_i)$, $i=1,2,\cdots,k$, and if $\bar{\xi}_1=\bar{\xi}_2=\cdots$*

$=\bar{\xi}_k=X$, *then there exists an element* ξ *of* $\bigcap_{i=1}^k \Sigma(f_i)$ *such that* $\bar{\xi}=X$.

Corollary 2. *If* $(X, f_i) \in \tilde{\Omega}$, $i=1, 2, \cdots, k$, *then there exists an element* ξ *of* $\bigcap_{i=1}^k \Sigma(f_i)$ *such that* $\bar{\xi}=X$.

Lemma 11. $(XY, f) \in \tilde{\Omega}$ *if and only if* $(X, Yf) \in \tilde{\Omega}$, *for any* $X, Y \in \bar{\Sigma}$ *and* $f \in \mathfrak{F}$.

Proof. Suppose that $(XY, f) \in \tilde{\Omega}$. For $\zeta \in \Sigma(f)$ such that $\bar{\zeta}=XY$, there exists $\xi \in \Sigma(Yf)$ such that $\bar{\xi}=X$ (Lemma 8), and this implies $(X, Yf) \in \tilde{\Omega}$. Conversely suppose that $(X, Yf) \in \tilde{\Omega}$. Then we have $\xi \in \Sigma(Yf)$ such that $\bar{\xi}=X$. Corollary to Lemma 7 implies that $Y\xi \in \Sigma(f)$. Since $\overline{Y\xi}=Y\bar{\xi}=XY$, we have $(XY, f) \in \tilde{\Omega}$.

Put $\tilde{\mathfrak{G}}(X)=\{f \mid (X, f) \in \tilde{\Omega}\}$ for each $X \in \bar{\Sigma}$. Then we have

Lemma 12. $\tilde{\mathfrak{G}}(X)$ *is a subgroup of* $\mathfrak{F}$ *for any* $X \in \bar{\Sigma}$.

Proof. It sufficies to show that $f-g \in \tilde{\mathfrak{G}}(X)$ for $f, g \in \tilde{\mathfrak{G}}(X)$. Corollary 2 to Lemma 10 implies that there is $\xi \in \Sigma(f) \cap \Sigma(g)$ such that $\bar{\xi}=X$. We have $\xi(n)(f-g)=\xi(n)f-\xi(n)g \in \mathfrak{G}$ for any n and thus we have $(X, f-g)=(\bar{\xi}, f-g) \in \tilde{\Omega}$. Hence $f-g \in \tilde{\mathfrak{G}}(X)$.

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