84. *The Additive Structure of the Unrestricted Z_p-Bordism Groups $\mathcal{O}_n(Z_p)$*

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Introduction. In this note we compute the additive structure of $\mathcal{O}_n(Z_p)$ and obtain that for $n \geqslant 0$,

$$\mathcal{O}_n(Z_p) \approx \begin{cases} \text{2-torsion} & \text{for } n \text{ odd,} \\ \text{free} + \text{2-torsion} & \text{for } n \text{ even,} \end{cases}$$

where the 2-torsion part consists of elements of order two.

We also compute the generators of $\mathcal{O}_n(Z_3)$ for $n \leqslant 7$, and study its connection with the Ω-module structure of $\mathcal{O}_*(Z_3)$ which we have determined in [5].

1. The additive structure of $\mathcal{O}_n(Z_p)$. We consider all (M^n, T) of Z_p-actions which form the Z_p-bordism group $\mathcal{O}_n(Z_p)$. First we shall need the exact sequence

$$0 \longrightarrow \Omega_n \xrightarrow{i_*} \mathcal{O}_n(Z_p) \xrightarrow{\nu} \mathfrak{M}_n(Z_p) \xrightarrow{\partial} \tilde{\Omega}_{n-1}(Z_p) \longrightarrow 0$$

which we already have in [5, Cororally 1.1]. Here $\tilde{\Omega}_{n-1}(Z_p)$ is the reduced, fixed point free, Z_p-bordism group, and $\mathfrak{M}_n(Z_p) = \sum_{k \geqslant 0} \Omega_{n-2k}(B(U(k_1) \times \cdots \times U(k_{p-1})))$, $k = k_1 + \cdots + k_{(p-1)/2}$. Moreover i_* is defined by $i_*[M^n] = [M \times Z_p, 1 \times \sigma] \in \mathcal{O}_n(Z_p)$ where σ is the map of period p which interchanges elements of Z_p; ν is defined by sending $[M^n, T] \in \mathcal{O}_n(Z_p)$ to the normal bundle over the fixed point set of T, $\sum_{k \geqslant 0} [\nu_k \to F_T^{n-2k}] \in \mathfrak{M}_n(Z_p)$, where $\nu_k \to F_T^{n-2k}$ is the complex k-dimensional normal bundle over the union F_T^{n-2k} of the $(n-2k)$-dimensional components of the fixed point set of T, and ∂ is defined by sending $\sum [V^{n-2k}, g_k] = \sum [\xi_k \to V^{n-2k}] \in \mathfrak{M}_n(Z_p)$ to the sphere bundles $\sum [S(\xi_k), \rho] \in \tilde{\Omega}_{n-1}(Z_p)$ where $\rho = \exp(2\pi i/p)$ and $\xi_k \to V^{n-2k}$ is the complex k-plane bundle classified by the map $g_k \colon V^{n-2k} \to B(U(k_1) \times \cdots \times U(k_{(p-1)/2}))$.

We also need several facts provided by Conner and Floyd in [3]:

For $X = B(U(k_1) \times \cdots \times U(k_{(p-1)/2}))$, $\Omega_n(X) \approx \sum_{j=0}^{n} H_j(X; \Omega_{n-j})$, [3, 15.2].

For a Ω-base $\{[S^{2i-1}, \rho]\}$ of $\tilde{\Omega}_*(Z_p)$, [3, 34.3], $[S^{2i-1}, \rho]$ has order p^{a+1} where $a(2p-2) < 2i-1 < (a+1)(2p-2)$, [3, 36.1].

And if $2i-1 = a(2p-2)+1$, then $p^a[S^{2i-1}, \rho] = b[S^1, \rho] \cdot [CP(p-1)]^a$ where $b \not\equiv 0 \pmod{p}$, [3, 36.2].

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For $n \geqslant 0$, $\tilde{\Omega}_{2n}(Z_p)=0$, [3, 34.2].

We may now show the following

Proposition. *For* $n \geqslant 0$,

$$\mathcal{O}_n(Z_p) \approx \begin{cases} 2\text{-}torsion & for\ n\ odd, \\ free + 2\text{-}torsion & for\ n\ even, \end{cases}$$

where the 2-*torsion part consists of elements of order two.*

Proof. Let the homomorphism $j_*: \Omega_n \to \mathcal{O}_n(Z_p)$ be defined by $j_*[M^n] = [M^n, 1] \in \mathcal{O}_n(Z_p)$. Then in the the following diagram

$$\begin{array}{ccccccccc} & & \Omega_{2n+1} \approx 2\text{-torsion} & & & & 0 & & \\ & & \downarrow j_* & & & & \| & & \\ 0 \to & \Omega_{2n+1} & \xrightarrow{i_*} \mathcal{O}_{2n+1}(Z_p) & \xrightarrow{\nu} & \sum \Omega_{2n+1-2k}(B(U(k_1)\times \cdots \times U(k_{p-1/2}))) & \xrightarrow{\partial} & \tilde{\Omega}_{2n}(Z_p) & \to 0, & \\ & \wr\wr & & & \wr\wr & & & & \\ & 2\text{-torsion} & & & \sum\sum H_{\text{even}}(B(U(k_1)\times \cdots \times U(k_{p-1/2}))\,;\,\Omega_{\text{odd}}) \approx 2\text{-torsion}, & & & & \end{array}$$

where $k=k_1+\cdots+k_{(p-1)/2}$, we see that $\mathcal{O}_{2n+1}(Z_p) \approx$ 2-torsion. The 2-torsion consists of elements of order two. For if we assume that $\mathcal{O}_{2n+1}(Z_p)$ has elements of order 4, then for some $[M^{2n+1}] \in \Omega_{2n+1}$ there is $[N^{2n+1}, T] \in \mathcal{O}_{2n+1}(Z_p)$ such that $i_*[M]=2[N,T]$. But in case it occurs, we may show that $[M]$ must vanish. To see this, we define $\varepsilon: \mathcal{O}_{2n+1}(Z_p) \to \Omega_{2n+1}$ by $\varepsilon[M,T]=[M]$ such that $\varepsilon j_*=1$. Then $\varepsilon i_*[M]=\varepsilon[M\times Z_p, 1\times\sigma]=[M \times Z_p]=p[M]$. On the other hand, $\varepsilon i_*[M]=\varepsilon(2[N,T])=2\varepsilon[N,T]=2[N]=0$. We thus have $p[M]=0$, and so $[M]=0$.

By the same arguments we obtain $\mathcal{O}_{2n}(Z) \approx$ free + 2-torsion, where the 2-torsion part also consists of order two. The assertion thus follows.

2. The generators of $\mathcal{O}_n(Z_3)$.

The explicit results of Thom groups for $n \leqslant 11$ are as follows.

$$\Omega_0=Z,\ \Omega_1=\Omega_2=\Omega_3=0,\ \Omega_4=Z,\ \Omega_5=Z_2,\ \Omega_6=\Omega_7=0,$$
$$\Omega_8=Z+Z,\ \Omega_9=Z_2+Z_2,\ \Omega_{10}=Z_2,\ \Omega_{11}=Z_2.$$

Let $P(m,n)=S^m\times CP(n)/\sim$ be the Dold manifold, where $(x,z) \sim(-x,\bar{z})$ for $x\in S^m$ and $z \in CP(n)$ with $\bar{z}$ its conjugate. $P(m,n)$ is orientable if and only if $m \not\equiv n \pmod 2$ or $m=0$. The generators of Ω_n for $n \leqslant 11$ are:

Ω_4: $[CP(2)]$, Ω_5: $[P(1,2)]$, Ω_8: $[CP(2)\times CP(2)]$, $[CP(4)]$,
Ω_9: $[P(1,4)]$, $[CP(2)\times P(1,2)]$, Ω_{10}: $[P(1,2)\times P(1,2)]$,
Ω_{11}: $[P(3,4)]$, [1, Theorem 2].

We compute the additive structure of $\mathcal{O}_n(Z_3)$ for $n\leqslant 11$ and obtain the explicit results in the following.

$$\mathcal{O}_0(Z_3)=Z+Z,\ \mathcal{O}_1(Z_3)=0,\ \mathcal{O}_2(Z_3)=Z,\ \mathcal{O}_3(Z_3)=0,\ \mathcal{O}_4(Z_3)=Z+Z+Z+Z,$$
$$\mathcal{O}_5(Z_3)=Z_2+Z_2,\ \mathcal{O}_6(Z_3)=Z+Z+Z+Z,\ \mathcal{O}_7(Z_3)=Z_2,\ \mathcal{O}_8(Z_3)=\underbrace{Z+\cdots+Z}_{11},$$
$$\mathcal{O}_9(Z_3)=\underbrace{Z_2+\cdots+Z_2}_{6},\ \mathcal{O}_{10}(Z_3)=(\underbrace{Z+\cdots+Z}_{12})+Z_2+Z_2,\ \mathcal{O}_{11}(Z_3)=\underbrace{Z_2+\cdots+Z_2}_{7}.$$

The generators of $\mathcal{O}_n(Z_3)$ for $n \leqslant 7$ are as follows.

(0) It is easy to see that $\mathcal{O}_0(Z_3) = j_*\Omega_0 + i_*\Omega_0$ where j_* and i_* are defined by $j_*[M] = [M, 1]$ and $i_*[M] = [M \times Z_3, 1 \times \sigma]$ for $[M] \in \Omega_0$. The map σ is a map of period 3 which interchanges elements of Z_3.

(1) $\mathcal{O}_1(Z_3) = 0$.

(2) $\mathcal{O}_2(Z_3)$ is generated by $[H, \tilde{T}]$ where H is an oriented differentiable 2-manifold with the fixed point set $F_{\tilde{T}}$ consisting of three points. Such a manifold can be constructed as follows. Define a curve $H \subset CP(2)$ by $H = \{[z_0, z_1, z_2] \mid z_0^3 + z_1^3 + z_2^3 = 0\}$ with an action $\tilde{T}$ given by $\tilde{T}([z_0, z_1, z_2]) = [z_0, z_1, \rho z_2]$, $\rho = \exp(2\pi i/3)$. Then $F_{\tilde{T}} = \{[-1, 1, 0], [-1, \rho, 0], [-1, \rho^2, 0]\}$, [2, p. 7]. This manifold H is a non-singular elliptic curve. In the following diagram

$$\begin{array}{ccccccccc} & & \Omega_2 = 0 & & & & & & \\ & & \downarrow j_* & & & & & & \\ 0 & \longrightarrow & \Omega_2 & \xrightarrow{i_*} & \mathcal{O}_2(Z_3) & \xrightarrow{\nu} & \mathfrak{M}_2(Z_3) & \xrightarrow{\partial} & \tilde{\Omega}_1(Z_3) \longrightarrow 0 \\ & & \| & & \wr\wr & & \| & & \| \\ & & 0 & & Z & & Z \approx \Omega_0(BU(1)) & & \{[S^1, \rho]\} \approx Z_3. \end{array}$$

We see that $[H, \tilde{T}]$, $F_{\tilde{T}} = \{3 \text{ points}\} \xrightarrow{\nu} [\nu_1 \to F_{\tilde{T}}] \xrightarrow{\partial} 3[S^1, \rho] = 0$.

(3) $\mathcal{O}_3(Z_3) = 0$.

(4) $\mathcal{O}_4(Z_3)$ is generated by $[CP(2), 1]$, $[CP(2) \times Z_3, 1 \times \sigma]$, $[CP(2), T_0]$ and $[CP(2), T_1]$ where $T_0([z_0, z_1, z_2]) = [\rho z_0, z_1, z_2]$ and $T_1([z_0, z_1, z_2]) = [\rho z_0, \rho^2 z_1, z_2]$ for $[z_0, z_1, z_2] \in CP(2)$. This may be seen in the following diagram.

$$\begin{array}{ccccccccc} & & \Omega_4 \approx Z & & & & & & \\ & & \downarrow j_* & & & & & & \\ 0 & \longrightarrow & \Omega_4 & \xrightarrow{i_*} & \mathcal{O}_4(Z_3) & \xrightarrow{\nu} & \mathfrak{M}_4(Z_3) & \xrightarrow{\partial} & \tilde{\Omega}_3(Z_3) \longrightarrow 0 \\ & & \wr\wr & & \wr\wr & & \| & & \| \\ & & Z & \text{i)} & Z & \longrightarrow & \Omega_4(BU(0)) \approx Z \longrightarrow 0 & & \{[S^3, \rho]\} \\ & & & & + & & + & & \wr\wr \\ & & & \text{ii)} & Z \longrightarrow 0 & & & & Z_3 \\ & & & & + & & \Omega_2(BU(1)) \approx Z & & \\ & & & \text{iii)} & Z & & + & \rightrightarrows & 0 \\ & & & & + & & \Omega_0(BU(2)) \approx Z & \longrightarrow & 0 \\ & & & \text{iv)} & Z & & & & \end{array}$$

i) $[CP(2)] \xrightarrow{j_*} [CP(2), 1] \xrightarrow{\nu} [\nu_0 \to CP(2)] \xrightarrow{\partial} 0$.

ii) $[CP(2)] \xrightarrow{i_*} [CP(2) \times Z_3, 1 \times \sigma] \xrightarrow{\nu} 0$. Here notice that $[CP(2) \times Z_3, 1 \times \sigma]$ is fixed point free.

iii) $[CP(2), T_0]$, $F_T = CP(1) \cup \{\text{a point}\} \xrightarrow{\nu} [\nu_1 \to CP(1)] + [\varepsilon^4 \to *] \xrightarrow{\partial} -[S^3, \rho] + [S^3, \rho] = 0$, where $\varepsilon^4 \to *$ is the trivial 4-plane bundle over the point $*$.

iv) $[CP(2), T_1]$, $F_{T_1} = \{3 \text{ points}\} \xrightarrow{\nu} 3[\varepsilon^4 \to *] \xrightarrow{\partial} 3[S^3, \rho] = 0$.

(5) $\mathcal{O}_5(Z_3)$ is generated by $[P(1, 2), 1]$ and $[P(1, 2) \times Z_3, 1 \times \sigma]$.

In the diagram

$$\begin{array}{ccccccccc} & & \Omega_5\approx Z_2 & & & & & & \\ & & \downarrow j_* & & & & & & \\ 0\longrightarrow & \Omega_5 & \xrightarrow{i_*} & \mathcal{O}_5(Z_3) & \xrightarrow{\nu} & \mathfrak{M}_5(Z_3) & \xrightarrow{\partial} & \tilde{\Omega}_4(Z_3) & \longrightarrow 0, \\ & \wr\wr & & & & \| & & \| & \\ & Z_2 & & & & \Omega_5(BU(0))\approx Z_2 & & 0 & \end{array}$$

i) $[P(1,2)]\xrightarrow{j_*}[P(1,2),1]\xrightarrow{\nu}[\nu_0\to P(1,2)]\xrightarrow{\partial}0.$

ii) $[P(1,2)]\xrightarrow{i_*}[P(1,2)\times Z_3, 1\times\sigma]\xrightarrow{\nu}0.$

(6) $\mathcal{O}_6(Z_3)$ is generated by $[CP(3), T_0 \mid T_0([z_0, z_1, z_2, z_3])=[\rho z_0, z_1, z_2, z_3]]$, $[{}^{\exists}M^6, T \mid F_T=CP(2)\cup\{3 \text{ points}\}]$, $[{}^{\exists}N^6, T' \mid F_{T'}=CP(1)\cup\{2 \text{ points}\}]$ and $[H,\tilde{T}]\cdot[CP(2), T_1]$ where $[H,\tilde{T}]$ is the 2-manifold stated in (2) and $T_1([z_0, z_1, z_2])=[\rho z_0, \rho^2 z_1, z_2]$ for $[z_0, z_1, z_2]\in CP(2)$.

For $\Omega_6=0$, $\tilde{\Omega}_5(Z_3)=\{[S^5,\rho]\}\approx Z_9$ and

$$\begin{aligned}\mathfrak{M}_6(Z_3)=\;&\Omega_4(BU(1))\approx H_4(BU(1);\Omega_0)+H_0(BU(1);\Omega_4)\approx Z+Z\\ &+\\ &\Omega_2(BU(2))\approx Z \qquad \text{[Cases: i), ii), iii) and iv)]}\\ &+\\ &\Omega_0(BU(3))\approx Z\end{aligned}$$

We have

i) $[CP(3), T_0]$, $F_{T_0}=CP(2)\cup\{\text{a point}\}\xrightarrow{\nu}[\nu_1\to CP(2)]+[\varepsilon^6\to *]\xrightarrow{\partial}0.$

ii) There is $[M^6, T]$ such that $F_T=CP(2)\cup\{3 \text{ points}\}$ with trivial normal bundle. $\xrightarrow{\nu}-[\varepsilon^2\to CP(2)]+3[\varepsilon^6\to *]\xrightarrow{\partial}0.$ For $[\varepsilon^2\to CP(2)]$ $\xrightarrow{\partial}[CP(2)\times S^1, 1\times\rho]=[CP(2)]\cdot[S^1,\rho]=3[S^5,\rho].$

iii) There is $[N^6, T']$ such that $F_{T'}=CP(1)\cup\{2 \text{ points}\}$ with $\nu[N^6, T']$ $=[i^*\gamma_1\oplus\varepsilon^2\to CP(1)]+2[\varepsilon^6\to *]$ where $i^*\gamma_1\oplus\varepsilon^2$ is obtained in the following:

$$\begin{array}{ccccc} i^*\gamma_1\oplus\varepsilon^2 & & \longrightarrow & & \gamma_2 \\ \downarrow & & & & \downarrow \\ CP(1) & \xrightarrow{i} & BU(1) & \longrightarrow & BU(2). \end{array}$$

Then $\partial\nu[N^6, T']=0$. For consider $[CP(2), T_0]$, $T_0([z_0, z_1, z_2])=[\rho z_0, z_1, z_2]$, we then have $\partial I_*\nu[CP(2), T_0]=[S^1,\rho]\cdot[CP(2)]=3[S^5,\rho]$ where I_*: $\Omega_{n-2k}(BU(k))\to\Omega_{n-2k}(BU(k+1))$ is a homomorphism induced by the homomorphism $U(k)\to U(k+1)$ sending the matrix α into $\begin{pmatrix}1 & 0\\ 0 & \alpha\end{pmatrix}$, [3, 38.6]. But $\partial I_*\nu[CP(2), T_0]=\partial\{[\nu_1\oplus\varepsilon^2\to CP(1)]+[\varepsilon^6\to *]\}=\partial[\nu_1\oplus\varepsilon^2\to CP(1)]+[S^5,\rho]$ which is $3[S^5,\rho]$. Hence $\partial[\nu_1\oplus\varepsilon^2\to CP(1)]=2[S^5,\rho]$ and ν_1 is conjugate to the bundle $i^*\gamma_1$.

iv) $[H,\tilde{T}]\cdot[CP(2), T_1]=[H\times CP(2), \tilde{T}\times T_1]$, $F_{\tilde{T}\times T_1}=\{3 \text{ pts.}\}\times\{3 \text{ pts.}\}\xrightarrow{\nu}$ $9[\varepsilon^6\to *]\xrightarrow{\partial}9[S^5,\rho]=0.$

(7) $\mathcal{O}_7(Z_3)$ is generated by $[{}^{\exists}V^7, T \mid F_T=P(1,2)$ with trivial normal

bundle].

Since $\Omega_7=\tilde{\Omega}_6(Z_3)=0$ and $\mathfrak{M}_7(Z_3)=\Omega_5(BU(1))\approx H_0(BU(1);\Omega_5)\approx Z_2$, $[V^7,T]$, $F_T=P(1,2)\xrightarrow{\nu}[\varepsilon^2\to P(1,2)]\xrightarrow{\partial}[P(1,2)\times S^1, 1\times\rho]=0$. There is then (W^7,T'), fixed point free, with $\partial(W^7,T')=(P(1,2)\times S^1,1\times\rho)$. We thus see that the generator $[V^7,T]$ is of the form $[(P(1,2)\times D^2)\cup W^7, 1\times\rho\cup T']$ where the two copies of $P(1,2)\times S^1$ are identified.

3. The Ω-module structure of $\mathcal{O}_*(Z_3)$. In [5, § 5] we have determined the Ω-module structure of $\mathcal{O}_*(Z_3)$. The result is as follows:

$$\mathcal{O}_*(Z_3)\approx\sum_{k\geqslant 0}\Omega\cdot\mu_k\oplus\sum_{l_0,\cdots,l_j\geqslant 0}\Omega\cdot\Gamma^{l_0}(\sigma_1^{l_1}\cdots\sigma_j^{l_j})$$

as free Ω-module, where $\sum\Omega\cdot\mu_k$ and $\sum\Omega\Gamma^{l_0}(\sigma_1^{l_1}\cdots\sigma_j^{l_j})$ are free Ω-modules generated by μ_k and $\Gamma^{l_0}(\sigma_1^{l_1}\cdots\sigma_j^{l_j})$ respectively which we shall explane in the following. In the exact sequence

$$0\longrightarrow\Omega_*\xrightarrow{i_*}\mathcal{O}_*(Z_3)\xrightarrow{\nu}\mathfrak{M}_*(Z_3)\xrightarrow{\partial}\tilde{\Omega}_*(Z_3)\longrightarrow 0,$$

there are closed oriented manifolds M^{4k}, $k=1,2,\cdots$, and $\beta_k\in\mathfrak{M}_*(Z_3)$ such that $\beta_k=3\theta_0^k+[M^4]\theta_0^{k-2}+[M^8]\theta_0^{k-4}+\cdots$, [5, § 5] where $\theta_0=[\varepsilon^2\to *]$ and that $\partial(\beta_k)=0$ in $\tilde{\Omega}_*(Z_3)$, [3, 46.1]. The generator μ_k is taken to be such an element of $\mathcal{O}_*(Z_3)$ that $\nu(\mu_k)=\beta_k$ for each $k\geqslant 1$ and $\mu_0=[Z_3,\sigma]$.

Let $\Omega_*(S^1)$ be the bordism group of free S^1-action and let $\mathcal{O}_*(S^1)$ and $\mathfrak{M}_*(S^1)$ be the bordism groups of semi-free S^1-actions which are just formed by replacing Z_3-actions by S^1-actions in $\Omega_*(Z_3)$,$\mathcal{O}_*(Z_3)$ and $\mathfrak{M}_*(Z_3)$ respectively. We shall use the Ω-module structure of $O_*(S^1)$ in that of $\mathcal{O}_*(Z_3)$, so consider now the diagram

$$\begin{array}{ccccccccc}
0\longrightarrow & \mathcal{O}_*(S^1) & \xrightarrow{\tilde{\nu}} & \mathfrak{M}_*(S^1) & \xrightarrow{\tilde{\partial}} & \Omega_*(S^1) & \longrightarrow 0\\
 & \downarrow\lambda & & =\downarrow\lambda & & \downarrow\lambda & \\
0\longrightarrow\Omega_*\xrightarrow{i_*} & {}_*\mathcal{O}(Z_3) & \xrightarrow{\nu} & \mathfrak{M}_*(Z_3) & \xrightarrow{\partial} & \tilde{\Omega}_*(Z_3) & \longrightarrow 0
\end{array}$$

where λ is the homomorphism defined by sending an S^1-action $[M,\tau]$ to a Z_3-action $[M,T]$; $\tilde{\nu}$ and $\tilde{\partial}$ are the homomorphisms quite analogous to ν and ∂. The first sequence is exact and $\mathfrak{M}_*(S^1)=\mathfrak{M}_*(Z_3)$, [4]. For any element $[M^n,\tau]\in\mathcal{O}_*(S^1)$, consider $(M\times D^2,1\times\tau_0)$ and $(M\times D^2,\tau\times\tau_0)$ where τ_0 is the usual S^1-action on D^2. Then $\partial(M\times D^2,1\times\tau_0)=(M\times S^1, 1\times\tau_0)$ and $\partial(M\times D^2,\tau\times\tau_0)=(M\times S^1,\tau\times\tau_0)$ are equivariantly diffeomorphic by an equivariant diffeomorphism $\varphi: M\times S^1\to M\times S^1$ defined by $\varphi(x,t)=(t(x),t)$. Form (M^{n+2},τ') from $(M\times D^2,1\times\tau_0)\cup(-M\times D^2,\tau\times\tau_0)$ by identifying $(M\times S^1,1\times\tau_0)$ and $(M\times S^1,\tau\times\tau_0)$ via φ. The Ω-map Γ: $\mathcal{O}_n(S^1)\to\mathcal{O}_{n+2}(S^1)$ is to be defined by $\Gamma[M^n,\tau]=[M^{n+2},\tau']$, and $\sigma_i=[CP(i+1),\tau]$, $\tau(t,[z_0,z_1,\cdots,z_{i+1}])=[tz_0,z_1,\cdots,z_{i+1}]$, $t\in S^1$. We then have

$$\mathcal{O}_*(S^1)\approx\sum\Omega\cdot\Gamma^{l_0}(\sigma_1^{l_1}\cdots\sigma_j^{l_j})$$

as free Ω-module, [4]. Here $\tilde{\nu}(\sigma_i)=\theta_i-\theta_0^{i+1}$ where $\theta_i=[\bar{\eta}\to CP(i)]$, $\bar{\eta}\to CP(i)$ is the complex line bundle over $CP(i)$ induced from the universal bundle over $BU(1)$ by the inclusion $i: CP(i) \to BU(1)$. We shall express $\mathcal{O}_n(Z_3)$ for $n \leqslant 7$ in the notations given above. With this expression, we may have a clearer sight of the Ω-module structure of $\mathcal{O}_*(Z_3)$ and its connection with that studied in § 2.

(0) $\mathcal{O}_0(Z_3) \approx \Omega_0 \cdot 1 + \Omega_0 \cdot \mu_0$. (1) $\mathcal{O}_1(Z_3) = 0$.

(2) $\mathcal{O}_2(Z_3) \approx \Omega_0 \cdot \mu_1$. (3) $\mathcal{O}_3(Z_3) = 0$.

(4) $\mathcal{O}_4(Z_3) \approx \Omega_4 \cdot 1 + \Omega_4 \cdot \mu_0 + \Omega_0 \cdot \sigma_1 + \Omega_0 \cdot \mu_2$.

(5) $\mathcal{O}_5(Z_3) \approx \Omega_5 \cdot 1 + \Omega_5 \cdot \mu_0$.

(6) $\mathcal{O}_6(Z_3) \approx \Omega_0 \cdot \sigma_2 + \Omega_0 \cdot \mu_3 + \Omega_0 \cdot \Gamma(\sigma_1) + \Omega_4 \cdot \mu_1$.

(7) $\mathcal{O}_7(Z_3) \approx \Omega_5 \cdot \mu_1$.

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