83. *Some Radii Associated with Polyharmonic Equation.* II

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Introduction. In the preceding paper [2], we treated the polyharmonic inner radius of a domain and in the present paper we are going to deal with the polyharmonic outer radius. G. Pólya and G. Szegö [3] defined the outer radius of a bounded domain by a conformal correspondence from the exterior of a given bounded domain to that of a circle and showed that it can be also given by the Green's function of the exterior of a bounded domain relative to the Laplace's equation $\Delta u=0$. Moreover defining the biharmonic outer radius of a domain by the Green's function of the exterior of it concerning with the biharmonic equation $\Delta^2 u=0$, they calculated the ordinary outer and biharmonic outer radii of a nearly circular domain. The aim of this paper is to extend the above results. In the first place, we obtain the Green's function of the exterior of a disk with the pole the point at infinity relative to the n-harmonic equation $\Delta^n u=0$ and define the n-harmonic outer radius of a bounded domain. Applying the above results, we compute the n-harmonic outer radius of a nearly circular domain and it is noticeable that it is monotonously increasing with respect to integer n, which is contrary to the fact in case of inner radius.

1. Outer radii associated with polyharmonic equations.

We use the following notations hereafter. Let D be a bounded and simply connected domain in the complex z-plane, C the boundary of D, $\tilde{D}$ the exterior of D, $z=x+iy$ the variable point in $\tilde{D}$, r the distance from the origin to z and ∞ the point at infinity of the extended complex plane.

Definition 1. The function satisfying following two conditions is called the Green's function of D with the pole ∞ relative to the n-harmonic equation $\Delta^n u=0$.

(i) The function has in a neighbourhood of ∞ the form excepting plus and minus signs

$$\log r+ar^{2(n-1)}+P(x, y)+h_n(z),$$

where the function $P(x,y)$ is a polynomial of x and y with order $\leqq 2n-3$ and $h_n(z)$ satisfies the equation $\Delta^n u=0$ in $\tilde{D}$.

(ii) On the boundary C, the function and all its normal derivatives of order $\leqq n-1$ vanish.

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Theorem 1. *If D is the disk $|z|<R$ in the complex z-plane, the Green's function $G_n(z)$ of D with the pole ∞ relative to the equation $\Delta^n u=0$ is as follows,*

$$G_n(z)=\log\frac{r}{R}+\frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{k}\left(1-\frac{r^2}{R^2}\right)^k.$$

Proof. It is obvious that the function $G_n(z)$ satisfies the condition (i) of the Green's function. Denoting

$$\lambda=\frac{r^2}{R^2},$$

we can rewrite the function $G_n(z)$ as

$$G_n(z)=\frac{1}{2}\left\{\log\lambda+\sum_{k=1}^{n-1}\frac{1}{k}(1-\lambda)^k\right\}.$$

And if $f(\lambda)$ denotes the following function

$$\log\lambda+\sum_{k=1}^{n-1}\frac{1}{k}(1-\lambda)^k,$$

$f(1)$ and $f^{(\alpha)}(1)$ for such an integer α as $1\leqq\alpha\leqq n-1$ vanish. Consequently we can prove that the function $G_n(z)$ satisfies the condition (ii) of the Green's function. That establishes the theorem.

G. Pólya and G. Szegö [3] defined the outer radius $\bar{r}$ of a given domain D as follows: $\tilde{D}$ being mapped conformally onto the exterior of a circle so that both points at infinity correspond each other and the linear magnification at ∞ is equal to 1, the radius of the circle so obtained is $\bar{r}$. When the Green's function of $\tilde{D}$ with the pole ∞ relative to the equation $\Delta u=0$ is

$$\log r-h_1(z),$$

they showed that the outer radius $\bar{r}$ is determined by

$$\log\bar{r}=\lim_{z\to\infty}h_1(z).$$

They also defined the biharmonic outer radius associated with the biharmonic equation $\Delta^2 u=0$ as follows: Denoted the Green's function of $\tilde{D}$ with the pole ∞ relative to the biharmonic equation $\Delta^2 u=0$ by

$$\log\frac{1}{r}+ar^2+bx+cy+h_2(z),$$

and putting

$$\frac{1}{2\bar{s}^2}=a,$$

the positive quantity $\bar{s}$ is called the biharmonic outer radius of D.

Now we define the n-harmonic outer radius of D associated with the n-harmonic equation $\Delta^n u=0$.

Definition 2. If the Green's function of a domain $\tilde{D}$ with the pole ∞ relative to the equation $\Delta^n u=0$ is

$$\log r+a\,r^{2(n-1)}+P(x,y)+h_n(z),$$

and we put

$$\log \bar{r}_1 = -\lim_{z\to\infty} h_1(z) \qquad (n=1),$$
$$\frac{1}{2(n-1)\bar{r}_n^{2(n-1)}} = |a| \qquad (n \geqq 2),$$

we call the positive quantity $\bar{r}_n$ the n-harmonic outer radius of the domain D.

Remark. When the domain D is a disk $|z|<R$ in the complex z-plane, the Green's function of $\tilde{D}$ with the pole ∞ relative to the equation $\Delta u=0$ is

$$\log \frac{r}{R},$$

and the Green's function of the same relative to the equation $\Delta^2 u=0$ has been given by G. Pólya and G. Szegö as follows

$$\log \frac{R}{r} - \frac{R^2-r^2}{2R^2}.$$

Using the preceding two Green's functions and the Green's function given in Theorem 1, we can obtain the ordinary outer radius, the biharmonic outer radius and the n-harmonic outer radius for an arbitrary integer $n(n\geqq 3)$ of the disk $|z|<R$, which are equal to the radius R of the given disk.

2. Outer radii of a nearly circular domain.

In this section, we treat the polyharmonic outer radius of a nearly circular domain defined in former section.

Definition 3. Let

$$r = 1+\rho(\varphi) \tag{1}$$

be the equation of the boundary of a domain in polar coordinate r and φ, where the periodic function $\rho(\varphi)$ represents the infinitesimal variation of a unit circle. We call the domain bounded by (1) the nearly circular domain.

We consider the Fourier series

$$\rho(\varphi) = a_0 + 2\sum_{m=1}^{+\infty} (a_m \cos m\varphi + b_m \sin m\varphi), \tag{2}$$

where each coefficient a_m or b_m is the infinitesimal of the first order. Terms of higher infinitesimal than the second order are neglected in all the discussions of this section.

G. Pólya and G. Szegö [3] obtained the ordinary outer radius $\bar{r}$ and the biharmonic outer radius $\bar{s}$ of the nearly circular domain as follows,

$$\begin{aligned} \bar{r} &= 1+a_0+\sum_{m=1}^{+\infty}(2m-1)(a_m^2+b_m^2), \\ \bar{s} &= 1+a_0+\sum_{m=1}^{+\infty}(4m-3)(a_m^2+b_m^2). \end{aligned} \tag{3}$$

As an extension of (3), we prove the following theorem.

Theorem 2. *For an arbitrary positive integer n, the n-harmonic outer radius $\bar{r}_n$ of the nearly circular domain $r<1+\rho(\varphi)$ is*

$$\bar{r}_n=1+a_0+\sum_{m=1}^{+\infty}(2nm-2n+1)(a_m^2+b_m^2). \tag{4}$$

Consequently, $\bar{r}_n$ increases monotonously with respect to n.

Proof. We seek the Green's function $G_n(z)$ of $r>1+\rho(\varphi)$ with the pole ∞ relative to the equation $\Delta^n u=0$ in the form

$$G_n(z)=\log r+\frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{k}(1-r^2)^k+p(r,\varphi)+q(r,\varphi),$$

$$p(r,\varphi)=\sum_{m=0}^{+\infty}\sum_{k=0}^{n-1}r^{2k-m}(A_{k,m}\cos m\varphi+B_{k,m}\sin m\varphi),$$

$$q(r,\varphi)=\sum_{m=0}^{+\infty}\sum_{k=0}^{n-1}r^{2k-m}(A'_{k,m}\cos m\varphi+B'_{k,m}\sin m\varphi),$$

where the coefficients of $p(r,\varphi)$ are of the first order and those of $q(r,\varphi)$ of the second order. The n-harmonic outer radius $\bar{r}_n$ is determined by

$$\frac{1}{2(n-1)\bar{r}_n^{2(n-1)}}=\left|\frac{(-1)^{n-1}}{2(n-1)}+A_{n-1,0}+A'_{n-1,0}\right|,$$

and so we have

$$\bar{r}_n=1+(-1)^n(A_{n-1,0}+A'_{n-1,0})+\frac{2n-1}{2}A^2_{n-1,0}. \tag{5}$$

Setting

$$\lambda=r^2 \text{ and } F(\lambda)=\frac{1}{2}\left\{\log\lambda+\sum_{k=1}^{n-1}\frac{1}{k}(1-\lambda)^k\right\},$$

we can rewrite as

$$G_n(z)=F(\lambda)+p(r,\varphi)+q(r,\varphi).$$

Let ν be the normal of the boundary of the nearly circular domain, then the condition $\partial^m G/\partial\nu^m=0$ on the boundary can be replaced by $\partial^m G/\partial r^m=0$. We obtain the following equality

$$\frac{dF}{dr}=\frac{1}{\lambda}(1-\lambda)^{n-1}r,$$

and neglecting the terms higher than the second order, on the boundary $r=1+\rho(\varphi)$, we have

$$F(\lambda)=0 \text{ and } \frac{d^\alpha F}{dr^\alpha}=0 \qquad 1\leqq\alpha\leqq n-3;$$

that is, $F(\lambda)$ and all its derivatives order $\leqq n-3$ are negligible on the boundary. So the boundary conditions are

$$p(1,\varphi)+\rho(\varphi)\frac{\partial}{\partial r}p(1,\varphi)+q(1,\varphi)=0,$$

$$\frac{\partial^\alpha}{dr^\alpha}p(1,\varphi)+\rho(\varphi)\frac{\partial^{\alpha+1}}{\partial r^{\alpha+1}}p(1,\varphi)+\frac{\partial^\alpha}{\partial r^\alpha}q(1,\varphi)=0 \qquad 1\leqq\alpha\leqq n-3,$$

$$\frac{\partial^{n-2}}{\partial r^{n-2}}p(1,\varphi)+\rho(\varphi)\frac{\partial^{n-2}}{\partial r^{n-1}}p(1,\varphi)+\frac{\partial^{n-2}}{\partial r^{n-2}}q(1,\varphi) \tag{6}$$

$$=(-1)^n 2^{n-2}(n-1)!\,\{\rho(\varphi)\}^2$$

$$\frac{\partial^{n-1}}{\partial r^{n-1}}p(1,\varphi)+\rho(\varphi)\frac{\partial^n}{\partial r^n}p(1,\varphi)+\frac{\partial^{n-1}}{\partial r^{n-1}}q(1,\varphi)$$

$$=(-1)^n 2^{n-1}(n-1)!\,\rho(\varphi)+(-1)^{n-1}2^{n-3}(n-3)n!\,\{\rho(\varphi)\}^2.$$

The first order terms yield

$$\begin{aligned} &p(1,\varphi)=0,\\ (7)\qquad &\frac{\partial^\alpha}{\partial r^\alpha}p(1,\varphi)=0 \qquad 1\leqq\alpha\leqq n-2,\\ &\frac{\partial^{n-1}}{\partial r^{n-1}}p(1,\varphi)=(-1)^n 2^{n-1}(n-1)!\,\rho(\varphi). \end{aligned}$$

Noting that, by the first and second conditions of (7), $p(r,\varphi)$ has the factor $(r^2-1)^{n-1}$, and on account of the last condition of (7), we obtain

$$(8)\qquad p(r,\varphi)=-(1-r^2)^{n-1}\left\{a_0+2\sum_{m=1}^{+\infty}r^{-m}(a_m\cos m\varphi+b_m\sin m\varphi)\right\},$$

in particular,

$$(9)\qquad A_{n-1,0}=(-1)^n a_0.$$

We consider the second order terms. By the first and second equalities of (6) and those of (7) we have

$$q(1,\varphi)=0 \text{ and } \frac{\partial^\alpha}{\partial r^\alpha}q(1,\varphi)=0 \qquad 1\leqq\alpha\leqq n-3,$$

so that it must be the form

$$(10)\qquad \sum_{k=0}^{n-1}r^{2k}A'_{k,0}=(r^2-1)^{n-2}(Ar^2+B),$$

where A and B are constants, and so we have

$$(11)\qquad A'_{n-1,0}=A.$$

Taking now the mean values of second order terms, we find

$$A+B=(-1)^{n-1}(n-1)\left\{a_0^2+2\sum_{m=1}^{+\infty}(a_m^2+b_m^2)\right\},$$

$$(n+2)A+(n-2)B=(-1)^{n-1}n(n+1)\left\{a_1+2\sum_{m=1}^{+\infty}(a_m^2+b_m^2)\right\}+(-1)^n 8n\sum_{m=1}^{+\infty}m(a_m^2+b_m^2),$$

and so we have

$$(12)\qquad A=(-1)^n\left\{-\frac{2n-1}{2}a_0^2+\sum_{m=1}^{+\infty}(2nm-2n+1)(a_m^2+b_m^2)\right\}.$$

By virtue of (5), (9), (11) and (12) we find

$$\bar{r}_n=1+a_0+\sum_{m=1}^{+\infty}(2nm-2n+1)(a_m^2+b_m^2).$$

This is the equality (4) of the theorem.

References

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