82. On Integral Representation Involving Meijer's G-Function

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The object of the present paper is to study the following integral relation of Meijer's G-function:

$$\int_0^\infty\int_0^\infty \frac{y^{2u-1}x^{2v-1}(x^2+y^2)}{(a^2y^2+b^2x^2)^{u+v}} G_{p,q}^{m,n}\left(\frac{\alpha x^2y^2(x^2+y^2)}{(a^2y^2+b^2x^2)^2}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right) f(x^2+y^2)dxdy$$
$$(1)\qquad =\frac{\sqrt{\pi}\,2^{-1-u-v}}{a^{2u}b^{2v}}\int_0^\infty G_{p+2,q+2}^{m,n+2}\left(\frac{\alpha z}{4}\,\middle|\,\begin{matrix}1-u,1-v,a_p\\ b_q,1-\frac{u}{2}-\frac{v}{2},\frac{1}{2}-\frac{u}{2}-\frac{v}{2}\end{matrix}\right)f(z)dz,$$

where $|\arg\alpha|<(m+n-1/2\,p-1/2\,q)\pi$, $R(u,v)>0$; $f(z)=0(z^{-\delta})$ for large z and $f(z)=0(z^{\varepsilon-1/2})$ for small z; $\delta>0$, $\varepsilon>0$.

Meijer's G-function is defined [1] by a Mellin-Barnes type integral:

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)=G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix}\right)$$
$$(2)\qquad =\frac{1}{2\pi i}\int_L \frac{\Gamma[(b_m)-s]\Gamma[1-(a_n)+s]}{\Gamma[1-(b_{m+1},q)+s]\Gamma[(a_{n+1},p)-s]}z^s\,dz,$$

where m,n,p,q are integers with $q\geq 1$; $0\leq n\leq p$, $0\leq m\leq q$, the parameters a_j and b_j are such that no poles of $\Gamma(b_j-s)$, $j=1,2,\cdots,m$ coincides with any pole of $\Gamma(1-a_j+s)$; $j=1,2,\cdots,n$. The poles of integrand must be simple and those of $\Gamma(b_j-s)$; $j=1,2,\cdots,m$ lie on one side of the contour L and those of $\Gamma(1-a_j+s)$; $j=1,2,\cdots,n$ must lie on the other side. The contour L runs from $-i^\infty$ to i^∞. Throughout the paper the above conditions shall be retained. The integral converges if $p+q<2(m+n)$ and $|\arg z|<(m+n-1/2\,p-1/2\,q)\pi$.

Then, by using the formula ([2], p. 377):

$$(3)\qquad \int_0^{\pi/2}\frac{\sin^{2u-1}\theta\cos^{2v-1}\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^{u+v}}d\theta=\frac{1}{2a^{2u}b^{2v}}\cdot\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)},\ R(u,v)>0,$$

we obtain our first result in the form

$$\int_0^{\pi/2}\frac{\sin^{2u-1}\theta\cos^{2v-1}\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^{u+v}}G_{p,q}^{m,n}\left(\frac{\alpha z\sin^2\theta\cos^2\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^2}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)d\theta$$
$$(4)\qquad =\frac{\sqrt{\pi}\,2^{-u-v}}{a^{2u}b^{2v}}G_{p+2,q+2}^{m,n+2}\left(\frac{\alpha z}{4}\,\middle|\,\begin{matrix}1-u,1-v,a_p\\ b_q,1-\frac{u}{2}-\frac{v}{2},\frac{1}{2}-\frac{u}{2}-\frac{v}{2}\end{matrix}\right),$$

provided $R(u)>0, R(v)>0, p+q<2(m+n)$ and $|\arg z|<(m+n-1/2\,p-1/2\,q)\pi$.

To prove (4), we substitute the contour integral (2) for the G-function and change the order of integration; then L.H.S. of (4) equals

$$\frac{1}{2\pi i}\int_L \frac{\Gamma[(b_m)-s]\Gamma[1-(a_n)+s)\alpha^s z^s}{\Gamma[1-(b_{m+1},q)+s]\Gamma[(a_{n+1},p)-s]} \times\int_0^{\pi/2} \frac{\sin^{2u+2s-1}\theta\cos^{2v+2s-1}\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^{u+v+2s}}d\theta\, ds \tag{5}$$

Now evaluating the inner integral with the help of (3) and using the definition (2) of G-function, we obtain the required result (4).

On putting $z=r^2$ in (4), and then multiplying both sides by $rf(r^2)$ and integrating between the limits $(0,\infty)$, we have

$$\begin{aligned}&\int_0^\infty rf(r^2)dr\int_0^{\pi/2}\frac{\sin^{2u-1}\theta\cos^{2v-1}\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^{u+v}} \\ &\quad\times G_{p,q}^{m,n}\left(\frac{\alpha r^2\sin^2\theta\cos^2\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^2}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)d\theta \\ &=\frac{\sqrt{\pi}\,2^{-u-v}}{a^{2u}b^{2v}}\int_0^\infty rf(r^2)G_{p+2,q+2}^{m,n+2}\left(\frac{\alpha r^2}{4}\,\middle|\,\begin{matrix}1-u,1-v,a_p\\ b_q,1-\frac{u}{2}-\frac{v}{2},\frac{1}{2}-\frac{u}{2}-\frac{v}{2}\end{matrix}\right)dr.\end{aligned} \tag{6}$$

Now on putting $x=r\cos\theta, y=r\sin\theta$ and after some simplification, we get the required result (1).

For applications, it is shown that the double integrals can be evaluated easily by choosing $f(z)$ in convenient form. That means, either the Right hand side integral of (1) is known after choosing $f(z)$ (see [2]) or can be evaluated.

Suppose $f(z)=z^\rho e^{-z}L_k^\sigma(z)$, ($L_k^\sigma(z)$: a Laguerre polynomial).

Then we have from relation (1):

$$\begin{aligned}&\int_0^\infty\int_0^\infty\frac{y^{2u-1}x^{2v-1}(x^2+y^2)^{\rho+1}}{(a^2y^2+b^2x^2)^{u+v}}e^{-x^2-y^2}L_k^\sigma(x^2+y^2)G_{p,q}^{m,n} \\ &\quad\times\left(\frac{\alpha x^2y^2(x^2+y^2)}{(a^2y^2+b^2x^2)^2}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)dxdy=\frac{\sqrt{\pi}\,2^{-1-u-v}}{a^{2u}b^{2v}} \\ &\quad\times\int_0^\infty G_{p+2,q+2}^{m,n}\left(\frac{\alpha z}{4}\,\middle|\,\begin{matrix}1-u,1-v,a_p\\ b_q,1-\frac{u}{2}-\frac{v}{2},\frac{1}{2}-\frac{u}{2}-\frac{v}{2}\end{matrix}\right)z^\rho e^{-z}L_k^\sigma(z)dz.\end{aligned} \tag{7}$$

By evaluating the integral on the right (see [2]), we get

$$\begin{aligned}&\int_0^\infty\int_0^\infty\frac{y^{2u-1}x^{2v-1}(x^2+y^2)^{\rho+1}}{(a^2y^2+b^2x^2)^{u+v}}e^{-x^2-y^2}L_k^\sigma(x^2+y^2)G_{p,q}^{m,n} \\ &\quad\times\left(\frac{\alpha x^2y^2(x^2+y^2)}{(a^2y^2+b^2x^2)^2}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)dxdy \\ &=\frac{(-1)^k}{k!}\,\frac{\sqrt{\pi}\,2^{-1-u-v}}{a^{2u}b^{2v}}G_{p+4,q+3}^{m,n+4}\left(\frac{\alpha}{4}\,\middle|\,\begin{matrix}-\rho,\sigma-\rho,1-u,1-v,a_p\\ b_q,1-\frac{u}{2}-\frac{v}{2},\frac{1}{2}-\frac{u}{2}-\frac{v}{2}\end{matrix}\right),\end{aligned} \tag{8}$$

where $R(\rho+1+b_j)>-1, j=1,2,\cdots,m;\ p+q<2(m+n)$,

$$|\arg \alpha| < \left(m+n-\frac{1}{2}\rho-\frac{1}{2}q\right)\pi.$$

In a similar fashion, many more interesting results could be obtained.

References

[1] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi: Higher Transcendental Functions, Vol. 1. McGraw-Hill, New York, pp. 207, 209, 215 (1953).

[2] I. S. Gradshteyn and I. M. Ryzhik: Table of Integrals; Series, and Products (English Edition) (1965).