109. *An Analogue of the Paley-Wiener Theorem for the Euclidean Motion Group*

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1. Introduction. The purpose of this paper is to prove an analogue of the Paley-Wiener theorem for the group G of the motions of the n-dimensional euclidean space.

Let $\hat{G}$ be the set of all equivalence classes of irreducible unitary representations of G. Let $L_2(G)$ (resp. $L_2(\hat{G})$) be the Hilbert space of all square integrable functions on G (resp. $\hat{G}$) with respect to the Haar measure (resp. the Plancherel measure). Then the Plancherel theorem states that the Fourier transform gives an isometry of $L_2(G)$ onto $L_2(\hat{G})$ (see § 2).

Let $C_c^\infty(G)$ be the space of all infinitely differentiable functions with compact support on G. By an analogue of the Paley-Wiener theorem we mean the characterization of the image of $C_c^\infty(G)$ by the Fourier transform.

As a number of articles ([1], [2], [4], [7]–[9] and etc.) indicate, in order to attack the problem one has to consider the Fourier-Laplace transforms of $C_c^\infty(G)$ which are (operator-valued) entire analytic functions "of exponential type" on a certain complex manifold. In general, $\hat{G}$ is not a C^∞ manifold but the space of all orbits in a real analytic manifold by actions of the "Weyl group" which gives equivalence relations. The Fourier-Laplace transform T_f of an element f of $C_c^\infty(G)$ is defined on the "complexification" of this real analytic manifold and satisfies certain functional equations derived from the actions of the Weyl group.

Detailed proofs will appear elsewhere.

2. Preliminaries. Let G be the group of motions of n-dimensional euclidean space $\boldsymbol{R}^n$. Then G is realized as the group of $(n+1)\times(n+1)$-matrices of the form $\begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix}$, $(k\in SO(n), x\in \boldsymbol{R}^n)$. Let K and H be the closed subgroups of the elements $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, $(k\in SO(n))$ and $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $(x\in \boldsymbol{R}^n)$, respectively. Then H is an abelian normal subgroup of G and G is the semidirect of H and K. We normalize the Harr measure dg on G such that $dg=dxdk$, where $dx=(2\pi)^{-n/2}dx_1\cdots dx_n$

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and dk is the normalized Haar measure on K. Let $\mathfrak{H}=L_2(K)$ be the Hilbert space of all square integrable functions on K. We denote by $\boldsymbol{B}(\mathfrak{H})$ the Banach space of all bounded linear operators on $\mathfrak{H}$.

If G_1 is a subgroup of G, we denote by $\hat{G}_1$ the set of all equivalence classes of irreducible unitary representations of G_1. For an irreducible unitary representation σ of G_1, we denote by $[\sigma]$ the equivalence class which contains σ. Denote by $\langle\ ,\ \rangle$ the euclidean inner product on $\boldsymbol{R}^n$. Then we can identify $\hat{H}$ with $\boldsymbol{R}^n$ so that the value of $\xi\in\hat{H}$ at $x\in H$ is $e^{i\langle\xi,x\rangle}$. For simplicity, we identify $k\in SO(n)$ with $\begin{pmatrix}k & 0\\ 0 & 1\end{pmatrix}\in K$ and $x\in\boldsymbol{R}^n$ with $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\in H$. Because H is normal, K acts on H, and therefore on $\hat{H}$ naturally: $\langle k\xi,x\rangle=\langle\xi,k^{-1}x\rangle$. Let K_ξ be the isotropy subgroup of K at $\xi\in\hat{H}$. If $\xi\neq 0$, K_ξ is isomorphic to $SO(n-1)$.

The irreducible unitary representations of G were enumerated and constructed by G. W. Mackey [6] and S. Itô [5] as follows. We fix $\xi\in\hat{H}$. Let χ_σ and d_σ be the character and the degree of $[\sigma]\in\hat{K}_\xi$, respectively. Let R be the right regular representation of K. If $\sigma(k)=(\sigma_{ij}(k))(1\leqq i,j\leqq d_\sigma)$, we put

$$P^\sigma=d_\sigma\int_{K_\xi}\overline{\chi_\sigma(m)}R_m d_\xi m$$

and

$$P_i^\sigma=d_\sigma\int_{K_\xi}\overline{\sigma_{ii}(m)}R_m d_\xi m,$$

where $d_\xi m$ is the normalized Haar measure on K_ξ. Then P^σ and P_i^σ are both orthogonal projections of $\mathfrak{H}$. Put $\mathfrak{H}^\sigma=P^\sigma\mathfrak{H}$ and $\mathfrak{H}_i^\sigma=P_i^\sigma\mathfrak{H}$. We denote by U^ξ the unitary representation of G induced by ξ, i.e. for $g=\begin{pmatrix}k & x\\ 0 & 1\end{pmatrix}\in G$

$$(U_g^\xi F)(u)=e^{i\langle\xi,u^{-1}x\rangle}F(k^{-1}u),\qquad (F\in\mathfrak{H},u\in K).$$

The subspaces $\mathfrak{H}_i^\sigma(1\leqq i\leqq d_\sigma)$ are invariant under U^ξ and the representations of G induced on $\mathfrak{H}_i^\sigma$ under U^ξ are equivalent for all $1\leqq i\leqq d_\sigma$. We fix one of them and denote it by $U^{\xi,\sigma}$. Two representations $U^{\xi,\sigma}$ and $U^{\xi'\sigma'}$ are equivalent if and only if there exists an element $k\in K$ such that $\xi'=k\xi$ and $[\sigma]=[\sigma'^k]$ where $\sigma'^k(m)=\sigma'(kmk^{-1})$, $(m\in K_\xi)$.

First we assume that $\xi\neq 0$. Then $U^{\xi,\sigma}$ is irreducible and every infinite dimensional irreducible unitary representation is equivalent to one of $U^{\xi,\sigma}$, $(\xi\neq 0,[\sigma]\in\hat{K}_\xi)$. Since $\mathfrak{H}=\bigoplus_{[\sigma]\in\hat{K}_\xi}\mathfrak{H}^\sigma$ and $\mathfrak{H}^\sigma=\bigoplus_{i=1}^{d_\sigma}\mathfrak{H}_i^\sigma$, we have

$$U^\xi\cong\bigoplus_{[\sigma]\in\hat{K}_\xi}(\underbrace{U^{\xi,\sigma}\oplus\cdots\oplus U^{\xi,\sigma}}_{d_\sigma\text{ times}}).\tag{2.1}$$

Next we assume that $\xi=0$. Then $U^{\xi,\sigma}$ is reducible and $K_\xi=K$. For any irreducible unitary representation σ of K we define a finite dimensional irreducible unitary representation U^σ of G by $U_g^\sigma=\sigma(k)$ where $g=\begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \in G$. Then we have $\underbrace{U^{0,\sigma}\cong U^\sigma \oplus \cdots \oplus U^\sigma}_{d_\sigma \text{ times}}$ and $U^0 \cong \bigoplus_{[\sigma]\in \hat{K}} U^{0,\sigma}$. Moreover every finite dimensional irreducible unitary representation of G is equivalent to one of U^σ, ($[\sigma] \in \hat{K}$).

We denote by $(\hat{G})_\infty$(resp. $(\hat{G})_0$) the set of all equivalence classes of infinite (resp. finite) dimensional irreducible unitary representations of G.

Let $\boldsymbol{R}_+$ be the set of all positive numbers and let M be the subgroup of the elements $\begin{pmatrix} 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(m \in SO(n-1))$. Then for any $\xi \in \hat{H}$ of the form ${}^t(a, 0, \cdots, 0)$, $a \in \boldsymbol{R}_+$, we have $K_\xi = M$. It follows from the above results that $(\hat{G})_\infty$ can be identified with $\boldsymbol{R}_+ \times \hat{M}$. It can be proved that the Plancherel measure of $(\hat{G})_0$ is zero and the Plancherel measure on $(\hat{G})_\infty$ is explicitly expressed as $(2/2^{n/2}\Gamma(2/n))a^{n-1}da \otimes d_\sigma$. For any $f \in C_c^\infty(G)$, we put

$$T_f(\xi, \sigma) = \int_G f(g) U_g^{\xi,\sigma} dg \qquad (\xi \neq 0, [\sigma] \in \hat{K}_\xi).$$

For $\xi = {}^t(a, 0, \cdots, 0)$, $(a \in \boldsymbol{R}_+)$, we write briefly $T_f(\xi, \sigma) = T_f(a, \sigma)$. Then the following Plancherel formula holds:

$$\int_G |f(g)|^2 dg = \frac{2}{2^{n/2}\Gamma(n/2)} \sum_{[\sigma]\in \hat{M}} d_\sigma \int_{\boldsymbol{R}+} \| T_f(a, \sigma) \|_2^2 a^{n-1} da, \tag{2.2}$$

where $\| \ \|_2$ denotes the Hilbert-Schmidt norm.

For any $f \in C_c^\infty(G)$ we put

$$T_f(\xi) = \int_G f(g) U_g^\xi dg.$$

The space, on which $T_f(\xi, \sigma)$ operates, depends not only on σ but also on ξ. However $T_f(\xi)$ is an operator on a fixed Hilbert space $\mathfrak{H}$, so that we can consider the $\boldsymbol{B}(\mathfrak{H})$-valued function T_f. We shall call T_f the Fourier transform of f. As above we write $T_f(\xi) = T_f(a)$ for $\xi = {}^t(a, 0, \cdots, 0)$, $(a \in \boldsymbol{R}_+)$. Then it follows from (2.1) and (2.2) that

$$\int_G |f(g)|^2 dg = \frac{2}{2^{n/2}\Gamma(n/2)} \int_{\boldsymbol{R}+} \| T_f(a) \|_2^2 a^{n-1} da.$$

3. Definition and some properties of the Fourier-Laplace transform. For each $\zeta \in \hat{H}^c (\cong \boldsymbol{C}^n)$ we define a bounded representation of G on $\mathfrak{H}$ by

$$(U_g^\zeta F)(u) = e^{i\langle \zeta, u^{-1}x \rangle} F(k^{-1}u), \qquad (F \in \mathfrak{H}, u \in K),$$

where $g = \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \in G$. For any $f \in C_c^\infty(G)$, put

$$T_f(\zeta) = \int_G f(g) U_g^\zeta dg.$$

Then T_f is a $\boldsymbol{B}(\mathfrak{H})$-valued function on $\hat{H}^c$. We call T_f the Fourier-Laplace transform of f.

Since K is compact, for each $f \in C_c^\infty(G)$ there exists a positive number a such that supp $(f) \subset \left\{\begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \in G ; |x| \leqq a, k \in K\right\}$. We denote by r_f the greatest lower bound of such a's. Throughout this section we assume that $f \in C_c^\infty(G)$ such that $r_f \leqq a$ for a fixed $a \in \boldsymbol{R}_+$.

Lemma 1. *There exists a constant $C \geqq 0$ depending only on f such that $\|T_f(\zeta)\| \leqq C \exp a|\mathrm{Im}\,\zeta|$.*

This lemma is easily verified using the Schwarz's inequality.

A $\boldsymbol{B}(\mathfrak{H})$-valued function T on $\hat{H}^c$ is called entire analytic if it is analytic at each point of $\hat{H}^c$ (for the definition of a Banach space valued analytic function, see [3(a)]). Then it is easy to see that T_f is entire analytic and that, for any $\zeta \in \hat{H}^c$, $T_f(\zeta)$ leaves the space $C^\infty(K)$ invariant.

We denote by λ (resp. μ) the representation of K on $C^\infty(G)$ defined by

$$\lambda(k)f(g) = f(k^{-1}g) \qquad (\text{resp. } \mu(k)f(g) = f(gk))$$

for $k \in K$ and $g \in G$. We also denote by λ and μ the corresponding representations of the universal enveloping algebra of the Lie algebra $\mathfrak{k}$ of K. We denote by Δ the Casimir operator of K (In case $n=2$, we put $\Delta = -X^2$ for a non-zero $X \in \mathfrak{k}$). For any polynomial function p on $\hat{H}^c$, we define a differential operator $p(D)$ on H by $p(D) = p(1/i \cdot \partial/\partial x_1, \cdots, 1/i \cdot \partial/\partial x_n)$. The following lemma is not difficult to prove but plays an important role.

Lemma 2. 1) *For any non-negative integers l and m we have $\Delta^l T_f(\zeta) \Delta^m = T\lambda(\Delta^l)\mu(\Delta^m)_f(\zeta)$, $(\zeta \in \hat{H}^c)$.*

2) *For any K-invariant polynomial function p on $\hat{H}^c$, we have $p(\zeta)T_f(\zeta) = T_{p^*(D)f}(\zeta)$, $(\zeta \in \hat{H}^c)$, where $p^*(\zeta) = p(-\zeta)$.*

From Lemma 1 and Lemma 2 we have

Proposition 1. *For any polynomial function p on $\hat{H}^c$ and for any non-negative integers l and m, there exists a constant $C_p^{l,m}$ such that*

$$\|p(\zeta)\Delta^l T_f(\zeta)\Delta^m\| \leqq C_p^{l,m} \exp a|\mathrm{Im}\,\zeta|.$$

Finally from the definition of T_f we have the following proposition (the functional equations for T_f).

Proposition 2. $T_f(k\zeta) = R_k T_f(\zeta) R_k^{-1}$ $(\zeta \in \hat{H}^c, k \in K)$.

4. The analogue of the Paley-Wiener theorem.

Theorem. A $\boldsymbol{B}(\mathfrak{H})$*-valued function T on $\hat{H}$ is the Fourier transform of $f \in C_c^\infty(G)$ such that $r_f \leqq a (a>0)$ if and only if it satisfies the following conditions:*

(I) *T can be extended to an entire analytic function on $\hat{H}^c$.*

(II) *For any $\zeta \in \hat{H}^c$, $T(\zeta)$ leaves the space $C^\infty(K)$ invariant. Moreover for any polynomial function p on $\hat{H}^c$ and for any non-negative integers l and m, there exists a constant $C_p^{l,m}$ such that*

$$\|p(\zeta)\Delta^l T(\zeta)\Delta^m\| \leqq C_p^{l,m} \exp a|\mathrm{Im}\,\zeta|.$$

(III) *For any* $k \in K$,

$$T(k\zeta) = R_k T(\zeta) R_k^{-1} \qquad (\zeta \in \hat{H}^c).$$

It is easy to see that the necessity of the theorem follows from the properties of the Fourier-Laplace transform which we mentioned in § 3.

In the following we shall give an outline of a proof of the sufficiency of the theorem. For the sake of brevity we assume that $n \geqq 3$. In case $n=2$ the same method is valid with a slight modification.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$. Denote by $\mathfrak{k}^c$ (resp. $\mathfrak{t}^c$) the complexification of $\mathfrak{k}$ (resp. $\mathfrak{t}$). Fix an order in the dual space of $\sqrt{-1}\ \mathfrak{t}$. Let P be the positive root system of $\mathfrak{k}^c$ with respect to $\mathfrak{t}^c$. Let $\mathfrak{F}$ be the set of all dominant integral forms. Then $\Lambda \in \mathfrak{F}$ is the highest weight of some irreducible unitary representation of K if and only if it is lifted to a unitary character of the Cartan subgroup. Let $\mathfrak{F}_0$ be the set of all such Λ's. For any $\Lambda \in \mathfrak{F}_0$ we denote by τ_Λ the irreducible unitary (matrix) representation of K with the highest weight Λ. Then the mapping $\Lambda \to \tau_\Lambda$ gives the bijection between $\mathfrak{F}_0$ and $\hat{K}$. Let d_Λ be the degree of τ_Λ. Then by the Peter-Weyl theorem we can choose a complete orthonormal basis $\{\Phi_j\}_{j \in J}$ of $\mathfrak{H}$, consisting of the matrix elements of irreducible unitary representations of K, i.e. $\Phi_j = \sqrt{d_\Lambda}(\tau_\Lambda)_{p,q}$ for some $\Lambda \in \mathfrak{F}_0$ and $p, q = 1, \cdots, d_\Lambda$. Let J_Λ be the set of j in J such that $\Phi_j = \sqrt{d_\Lambda}(\tau_\Lambda)_{p,q}$ for some $p, q = 1, \cdots, d_\Lambda$. Let $(\ ,\ )$ be the inner product on the dual space of $\sqrt{-1}\ \mathfrak{t}$ induced by the Killing form and put $|\Lambda| = (\Lambda, \Lambda)^{1/2}$. We put as usual $\rho = \frac{1}{2} \sum_{\alpha \in P} \alpha$.

Let us now assume that T is an arbitrary $\boldsymbol{B}(\mathfrak{H})$-valued function on $\hat{H}$ satisfying the conditions (I)~(III) in the theorem. We define the kernel function of $T(\zeta)(\zeta \in \hat{H}^c)$ by

$$K(\zeta\,; u, v) = \sum_{i,j \in J} (T(\zeta)\Phi_j, \Phi_i)\Phi_i(u)\overline{\Phi_j(v)}, \ (u, v \in K) \tag{4.1}$$

Lemma 3. *For any* $\zeta \in \hat{H}^c$ *the series* $\sum_{i,j \in J} |(T(\zeta)\Phi_j, \Phi_i)|$ *converges, so that the series of the right side of* (4.1) *is absolutely convergent and moreover it is uniformly convergent on every compact subset of* $\hat{H}^c \times K \times K$.

For the proof of this lemma we use the condition (II) and the following facts: For every $\Lambda \in \mathfrak{F}_0$ and $j \in J_\Lambda$, we have $(\Delta + |\rho|^2)\Phi_j = |\Lambda + \rho|^2 \Phi_j$ and the Weyl's dimension formula $d_\Lambda = \prod_{\alpha \in P} (\Lambda + \rho, \alpha) / \prod_{\alpha \in P} (\rho, \alpha)$. And moreover, the Dirichlet series $\sum_{\Lambda \in \mathfrak{F}_0} |\Lambda + \rho|^{-s}$ converges if $s > [n/2]$ (see [3(b)]).

From Lemma 3 we have the following

Corollary. *The function* $\hat{H}^c \times K \times K \ni (\zeta, u, v) \to K(\zeta\,; u, v)$ *is of* C^∞ *class and entire analytic with respect to* ζ.

The condition (II) and the above mentioned facts imply also the following

Lemma 4. *For any polynomial function p on $\hat{H}^c$, there exists a constant C_p such that*

$$|p(\zeta)K(\zeta\,;u,v)|\leqq C_p \exp a|\mathrm{Im}\,\zeta|, \qquad (\zeta\in\hat{H}^c, u, v\in K).$$

Remark. $K(\zeta\,;u,v)$ is rapidly decreasing on the real axis $\hat{H}$.

Notice that from Lemma 3 the operator $T(\zeta)$ is of trace class (see [3(b)], Lemma 1). Now we define a function f on G by

$$f(g)=\frac{2}{2^{n/2}\Gamma(n/2)}\int_{\boldsymbol{R}^+} Tr(T(a)U^a_{g^{-1}})a^{n-1}da. \tag{4.2}$$

By the condition (III), we can prove that

$$K(k\zeta\,;u,v)=K(\zeta\,;uk^{-1},vk^{-1}) \tag{4.3}$$

for every $\zeta\in\hat{H}^c$ and $u,v,k\in K$. The formulae (4.2) and (4.3) imply

$$f\begin{pmatrix}k & x\\ 0 & 1\end{pmatrix}=\int_{\hat{H}}K(\xi\,;1,k^{-1})e^{-i\langle\xi,x\rangle}d\xi$$

$(k\in K, x\in \boldsymbol{R}^n)$ where $d\xi=(2\pi)^{-n/2}d\xi_1\cdots d\xi_n$.

It follows from Corollary to Lemma 3 and the above remark that f is of class C^∞. Making use of the classical Paley-Wiener theorem, from Lemma 4 we can prove that if $|x|>a$, $f\begin{pmatrix}k & x\\ 0 & 1\end{pmatrix}=0$ for any $k\in K$.

Our final task is to check that $T_f=T$, which can be shown by comparing the kernel functions of both operators.

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