102. On An Ergodic Abelian $\mathcal{M}$-Group*)

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Let $\mathcal{M}$ be an abelian von Neumann algebra, F an $\mathcal{M}$-group (i.e. a group of automorphisms of $\mathcal{M}$). Let $[F]$ denote the full group generated by F. Choda proved in [1] that F is maximal abelian in $[F]$ if F is ergodic, abelian and free, by techniques of cross product algebras. In this note we prove, by completely different techniques, the following theorem.

Theorem. *Suppose that $\mathcal{M}$ is an abelian von Neumann algebra, and F is an ergodic abelian $\mathcal{M}$-group.*
Then:

(i) F is free.

(ii) F is maximal abelian in $[F]$.

(iii) $F' \cap [F] = F$.

(iv) $\beta \in F' \Rightarrow E(\beta, \alpha) \neq 0$ for at most one $\alpha \in F$, where $E(\beta, \alpha)$ is by definition $\sup\{F$ projection in $\mathcal{M}$: $\beta(M) = \alpha(M)$ for all $M \in \mathcal{M}$ with $FM = M\}$.

Before we prove the preceding theorem, we shall prove an auxiliary result.

Lemma 1. *Suppose that $\mathcal{M}$ is an abelian von Neumann algebra, and F is an ergodic abelian $\mathcal{M}$-group. Suppose that β is in F'. Then if α_1 and α_2 are in F with $E(\beta, \alpha_1) \neq 0$, and $E(\beta, \alpha_2) \neq 0$, we have:*

$$E(\beta, \alpha_1) = E(\beta, \alpha_2).$$

Proof. Let β agree with α_i on a non-zero projection P_i of $\mathcal{M}(i=1, 2)$. Since F is ergodic there exists $\alpha \in F$ such that $Q = \alpha(P_1)P_2 \neq 0$. Now if $M \in \mathcal{M}$ with $\alpha(M)Q = \alpha(M)$ then $\beta(M) = \alpha_1(M)$. So for $M \in \mathcal{M}$ with $MQ = M$ we have first $\beta(M) = \alpha_2(M)$, and secondly $\beta(M) = (\alpha\beta) \times (\alpha^{-1}(M)) = \alpha\alpha_1(\alpha^{-1}(M)) = \alpha_1(M)$, where we have used both that $\beta \in F'$ and that F is abelian. Thus we see that α_1 and α_2 agree on $\alpha(P_1)P_2$. That is, any non-zero projection (of $\mathcal{M}$) on which β agrees with α_2 majorizes a non-zero projection (of $\mathcal{M}$) on which α_1 agrees with α_2. Therefore $E(\beta, \alpha_2)[I - E(\alpha_1, \alpha_2)] = 0$, or $E(\beta, \alpha_2) \leq E(\alpha_1, \alpha_2)$. By the definition of $E(\alpha_1, \alpha_2)$ we obtain

$$E(\beta, \alpha_2) \leq E(\beta, \alpha_1).$$

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The reverse inequality is obtained by reversing the roles of α_1 and α_2, and we conclude that

$$E(\beta, \alpha_1)=E(\beta, \alpha_2).$$

We shall also need the following result of Bures [2].

Lemma 2 [2, Proposition 4.3]. *Suppose that α and β are automorphisms of an abelian von Neumann algebra $\mathcal{M}$. Then there exists a family (E_i) of projections of $\mathcal{M}$ such that*

$$\Sigma E_i=I-E(\alpha, \beta)$$

and

$$(\alpha(E_i))(\beta(E_i))=0 \quad \textit{for each } i.$$

Now we prove our theorem.

Proof of Theorem. Ad (i). Let e be the identity of F, and let $\beta \in F \setminus \{e\}$. Since $E(\beta, \beta)=I$ and $\beta \neq e$, we have $E(\beta, e) \neq E(\beta, \beta)$. Now as F is abelian, $\beta \in F'$ and so, by Lemma 1, $E(\beta, e)=0$. By Lemma 2, there exists a family (E_i) of projections of $\mathcal{M}$ such that $\Sigma E_i=I$ and $\beta(E_i)E_i=0$ for each i. So F is free.

Ad (ii). Let F_1 be an abelian subset of $[F]$ containing F. Let $\beta \in F_1$. Then as F_1 is abelian and $F_1 \supset F$, $\beta \in F'$. Now $\beta \in [F]$ also, so

$$\sup \{E(\beta, \alpha) : \alpha \in F\}=I,$$

or

$$\sup \{E(\beta, \alpha) : \alpha \in F \quad \text{and} \quad E(\beta, \alpha) \neq 0\}=I.$$

By Lemma 1 this means that for some $\alpha_0 \in F$,

$$E(\beta, \alpha_0)=I \text{ i.e. } \beta=\alpha_0.$$

So $\beta \in F$. We conclude that $F_1=F$. Thus F is maximal abelian in $[F]$.

Ad (iii). As F is abelian we obviously have $F' \cap [F] \supset F$. The above proof of (ii) shows in fact that $F' \cap [F] \subset F$. Thus we have $F' \cap [F]=F$.

Ad (iv). Suppose that $E(\beta, \alpha_1) \neq 0$ and $E(\beta, \beta_2) \neq 0$ for α_1 and α_2 in F. Then by Lemma 1, α_1 and α_2 agree on the non-zero projection $Q \equiv E(\beta, \alpha_1)=E(\beta, \alpha_2)$. Now let (E_i) be any family of orthogonal projections in $\mathcal{M}$ such that $\alpha_1^{-1}\alpha_2(E_i)E_i=0$ for each i. Let $Q_i=QE_i$. Then we have $Q_i \leq Q$ and $Q_i \leq E_i$ so that $Q_i=\alpha_1^{-1}\alpha_2(Q_i)Q_i \leq \alpha_1^{-1}\alpha_2(E_i)E_i=0$ for each i. As $Q_i=QE_i$ and $Q \neq 0$, so $\Sigma E_i \neq I$. Now by (i), F is free. Thus $\alpha_1^{-1}\alpha_2=e$, i.e. $\alpha_1=\alpha_2$. This completes the proof.

References

[1] M. Choda and H. Choda: On extensions of automorphisms of abelian von Neumann algebras. Proc. Japan Acad., **43**, 295–299 (1967).

[2] D. Bures: Abelian subalgebras of von Neumann algebras. pre-print (to be published in the Memoirs of Amer. Math. Soc.).