70. *On the Integral of Cauchy-Stieltjes Type and I. I. Privalov's Fundamental Lemma. I*

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1. Introduction. The object of this note is to sketch some results on the boundary behaviour of the integral of Cauchy-Stieltjes type, obtained by the systematic use of generalized I. I. Privalov's fundamental lemmas. The details of proofs will be published in another journal in near future.

We call next integral as the integral of Cauchy-Stieltjes type

$$f(z)=\frac{1}{2\pi i}\cdot\int_L \frac{e^{i\varphi}dF(s)}{x-z}=\frac{1}{2\pi i}\int_0^l \frac{e^{i\varphi(s)}dF(s)}{x(s)-z}, \tag{1.1}$$

where L is a closed rectifiable Jordan curve of length l, s the arc length, $\varphi(s)$ the angle between the positive real axis and the tangent at the point $x(s)$ on L, and $F(s)$ the complex-valued function of s of bounded variation on the segment $[0, l]$. If $f(z)\equiv 0$ for z outside L, (1.1) is called Cauchy-Stieltjes integral ([3] p. 154).

Let x_0 be the point $x(s_0)$, L_ε the part of L which is left after cutting off the small arc with end points $x(s_0-\varepsilon)$ and $x(s_0+\varepsilon)$. If the next limit exists:

$$\lim_{\varepsilon\to+0}\frac{1}{2\pi i}\int_{L_\varepsilon}\frac{e^{i\varphi}dF(s)}{x-x_0},$$

then we call it the singular integral at x_0, and we denote it by

$$\frac{1}{2\pi i}\int_L \frac{e^{i\varphi}dF(s)}{x-x_0}. \tag{1.2}$$

In the distance ε from the point x_0, we choose a point z on the straight line zx_0 inclined by angle ψ_0 to the normal, i.e. $z=x_0\pm\varepsilon i e^{i(\varphi_0+\psi_0)}$, and we consider the difference:

$$F(\varepsilon, x_0, \psi_0)=\frac{1}{2\pi i}\left[\int_L \frac{e^{i\varphi}dF(s)}{x-z}-\int_{L_\varepsilon}\frac{e^{i\varphi}dF(s)}{x-x_0}\right],$$

which is well defined at the point x_0 on L where a definite tangent exists. Then I. I. Privalov's fundamental lemma ([3] p. 131) is as follows; *If $F'(s_0)$ exists, then the difference $F(\varepsilon,x_0,\psi_0)$ tends to $+\frac{1}{2}F'(s_0)$ $\left(-\frac{1}{2}F'(s_0)\right)$ uniformly with respect to ψ_0, $|\psi_0|\leqq\frac{\pi}{2}\cdot\theta$ $(0<\theta<1)$, when z tends to x_0 inside (outside) L respectively.*

2. Fundamental Lemma 1. We extend I. I. Privalov's lemma to the case where L has a corner at x_0 and both of $F'_{\pm}(s_0)$ exist.

Fundamental Lemma 1. *Suppose that there exists a corner at* x_0:

$$\varphi_1=\lim_{h\to+0}\varphi(s_0-h),\quad \varphi_2=\lim_{h\to+0}(s_0+h),\quad \theta=\varphi_2-\varphi_1 \qquad (|\theta|<\pi),$$

and further that $F(s)$ *is continuous at* $s=s_0$ *and has one-sided derivatives*: $a=F'_+(s_0)$, $b=F'_-(s_0)$. *Put*

$$z=x_0+\varepsilon e^{i\varphi_1}\cdot e^{i\alpha}=x_0+\varepsilon e^{i\varphi_2}\cdot e^{i(\alpha-\theta)} \qquad \text{for } \varepsilon>0, |\alpha|<\pi,$$

where

(2.1) $\quad |\cos\alpha|\leqq q, |\cos(\alpha-\theta)|\leqq q \qquad$ for a fixed q $(0<q<1)$.

Then we have

(2.2) $$f(z)-\frac{1}{2\pi i}\cdot\int_{L_\varepsilon}\frac{e^{i\varphi}dF(s)}{x-x_0}=\frac{a}{2\pi}(\theta-\alpha)+\frac{b}{2\pi}\alpha-\frac{a}{2}\cdot\text{sign}(\theta-\alpha)+o(1)$$

as $\varepsilon\to+0$ *uniformly with respect to* α *satisfying* (2.1). *Moreover,* (2.2) *still holds in the case that* $z=x_0+\varepsilon e^{i\varphi_1}\cdot e^{i\alpha(\varepsilon)}$ *and* $\alpha(\varepsilon)\to\alpha$ *as* $\varepsilon\to+0$.

By this lemma, we can extend considerably Z. Ditzian's theorems ([2]). As its second application, we can generalize to the Jordan domain classical theorems due to P. Fatou and A. Plessner.

Theorem 1. *Suppose that there exists a definite tangent at* $x_0=x(s_0)$ *and further that* $F(s)$ *is continuous at* $s=s_0$, *and has one-sided derivatives*: $a=F'_+(s_0)$, $b=F'_-(s_0)$. *Put*

$$z=x_0+\varepsilon e^{i\varphi_0}\cdot e^{i\alpha}, \qquad z^*=x_0+\varepsilon^* e^{i\varphi_0}\cdot e^{-i\alpha^*},$$
$$(\varphi_0=\varphi(s_0), 0<\alpha<\pi, 0<\alpha^*<\pi),$$

where

(1) $|\cos\alpha|\leqq q$ *for a fixed* q $(0<q<1)$,

(2) $\varepsilon^*\to0$, $\alpha^*\to\alpha$ *as* $\varepsilon\to+0$ *in such a manner that* $\varepsilon^*e^{-i\alpha^*}=\varepsilon e^{-i\alpha}(1+0(\varepsilon))$. *Under these conditions,*

$$f(z)-f(z^*)\to\left(1-\frac{\alpha}{\pi}\right)a+\frac{\alpha}{\pi}b,$$

$$f(z)+f(z^*)-\frac{1}{\pi i}\int_{L_\varepsilon}\frac{e^{i\varphi}dF(s)}{x-x_0}\to0$$

as $\varepsilon\to+0$ *uniformly with respect to* α *satisfying* (1).

Applying Theorem 1 to the unit disk, we can prove

Corollary 1. *Suppose that* $F(s)$ *is continuous at* $s=s_0$, *and has one-sided derivatives*: $a=F'_+(s_0)$, $b=F'_-(s_0)$. *Put*

$$z=re^{i\theta}=e^{is_0}+\varepsilon e^{i\varphi_0}, e^{i\alpha},$$

where $0<r<1, \varphi_0=s_0+\pi/2, 0<\alpha<\pi$. *Then following propositions hold*;

(1) $\frac{1}{2\pi}\cdot\int_0^{2\pi}\frac{1-|z|^2}{|e^{is}-z|^2}dF(s)\to\left(1-\frac{\alpha}{\pi}\right)a+\frac{\alpha}{\pi}b$ *as* $\varepsilon\to+0$ *uniformly with respect to* α *with* $|\cos\alpha|\leqq q<1$.

(2) $$-\frac{1}{\pi}\cdot\int_0^{2\pi}\frac{r\sin(s-\theta)}{|e^{is}-z|^2}dF(s)$$
$$+\frac{1}{2\pi}\int_\varepsilon^\pi\cot\left(\frac{t}{2}\right)d(F(s_0+t)+F(s_0-t))\to0$$

as $\varepsilon \to +0$ *uniformly with respect to* α *satisfying* $|\cos\alpha| \leqq q < 1$.

(3) *If* $a \neq b$, *then*

$$-\frac{1}{\pi}\cdot\int_0^{2\pi}\frac{r\sin(s-\theta)}{|e^{is}-z|^2}dF(s) \sim \log\left(\frac{1}{\varepsilon}\right)\cdot\frac{b-a}{\pi}$$

as $\varepsilon \to +0$ *uniformly with respect to* α *such that* $|\cos\alpha| \leqq q < 1$.