91. *On Hypersurfaces which are Close to Spheres*

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0. Some characterizations of the sphere among the closed strictly convex hypersurfaces in R^{n+1} were given in [1].
In particular, the following theorem holds:
A closed strictly convex hypersurface with $K_{n-1}/K_n=r$ is a hypersphere of radius r, where K_{n-1} is the $(n-1)$-th mean curvature and K_n is the Gaussian curvature.
Then, we prove

Theorem. *Let M be a closed strictly convex hypersurface in $R^{n+1}(n\geqslant 2)$. If the function K_{n-1}/K_n on M is sufficiently close to r, then M is arbitrary close to a hypersphere of radius r in the sense that it can be enclosed between two concentric hyperspheres whose radius is arbitrarily close to r.*
For the case where $n=2$, D. Koutroufiotis proved in [3]. Our proof of theorem is the same method of his proof in [3].

1. For the sake of simplicity, we shall assume our manifolds and mappings to be of class C^∞.
Let R^{n+1} be the $(n+1)$-dimensional euclidean space.
By a hypersurface in R^{n+1} we mean a n-dimensional connected manifold M with an immersion x.
Suppose M to be oriented. Then to $p\in M$, there is a uniquely determined unit normal vector $\xi(p)$ at $x(p)$.
We put

$$\mathrm{I}=dx\cdot dx, \qquad \mathrm{II}=-d\xi\cdot dx.$$

Let $k_1,\cdots,k_n$, are called the principal curvatures, be the eigenvalues of II relative to I. The i-th mean curvature K_i $(1\leqslant i\leqslant n)$ is given by the i-th elementary symmetric function divided by $\binom{n}{i}=n!/i!(n-i)!$ i.e.,

$$\binom{n}{i}K_i=\sum k_1\cdots k_i.$$

In particular, $K_n=k_1\cdots k_n$ is called the Gaussian curvature. We shall consider closed strictly convex hypersurfaces i.e., compact hypersurfaces for which the Gaussian curvature K_n never vanishes on M.

We shall assume that the normal vector ξ is interior. Let S^n be the unit sphere in R^{n+1}. We denote by g the induced Riemannian metric on S^n.

Since the Gaussian curvature K_n never vanishes on M, the spherical mapping ξ of M onto S^n is a diffeomorphism.

$$S^n \xrightarrow[\xi^{-1}]{} M \xrightarrow[x]{} R^{n+1}.$$

We put

$$X = x \circ \xi^{-1}.$$

We now remark that the i-th mean curvature $\tilde{K}_i$ of the hypersurface (S^n, X) is given by

$$\tilde{K}_i(\nu) = K_i(\xi^{-1}(\nu)) \qquad \text{at each point } \nu \in S^n.$$

We shall denote $\tilde{K}_i(\nu)$ by the same letter $K_i(\nu)$.

The support function φ of the hypersurface (S^n, X) is defined by

$$\varphi(\nu) = -X(\nu) \cdot \nu$$

where $\cdot$ is the inner product in R^{n+1}.

Then the support function φ satisfies the following differential equation:

(1.1) $$\Delta\varphi + n\varphi = nK_{n-1}/K_n,$$

where Δ is the Laplace-Beltrami operator with respect to the natural Riemannian metric g on S.

In fact, let $\{X_1, \cdots, X_n\}$ be an orthonormal basis in $T_\nu(S^n)$ and H be the symmetric tensor field of type (1,1) corresponding to the second fundamental form II.

We have

$$\begin{aligned}\Delta\varphi &= \sum_{i=1}^{n} \nabla_{X_i}\nabla_{X_i}\varphi = -\sum \nabla_{X_i}X \cdot \nabla_{X_i}\nu - X \cdot \sum \nabla_{X_i}\nabla_{X_i}\nu \\ &= \sum \nabla_{H^{-1}X_i}\nu \cdot \nabla_{X_i}\nu - X \cdot \Delta\nu = \sum g(H^{-1}X_i, X_i) + nX \cdot \nu \\ &= \text{Trace } H^{-1} - n\varphi = nK_{n-1}/K_n - n\varphi.\end{aligned}$$

Let U_1 and U_2 be open subsets of S^n defined by

$$U_1 = \left\{(x_1, \cdots, x_{n+1}) \in S^n \mid x_{n+1} > -\frac{1}{2}\right\},$$

$$U_2 = \left\{(x_1, \cdots, x_{n+1}) \in S^n \mid x_{n+1} < \frac{1}{2}\right\}.$$

Those open sets define an open covering of S^n and are coordinate neibourhoods with local coordinates $(y_1, \cdots, y_n)$.

Next, we shall define the some norms of functions on S^n.

The norm of a continuous function f on S is defined by

$$\|f\| = \sup_{\nu \in S^n} |f(\nu)|.$$

For some p, $1 < p < \infty$, and some integer k, the norm of a C^k-function f on S^n is defined by

$$\|f\|_{k,p} = \left\{\int\!\!\int_{U_1} \sum_{|\alpha| \leqslant k} |D^\alpha f|^p \, dU_1\right\}^{1/p} + \left\{\int\!\!\int_{U_2} \sum_{|\alpha| \leqslant k} |D^\alpha f|^p \, dU_2\right\}^{1/p},$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $D^\alpha f = \partial^{|\alpha|} f / \partial y_1^{\alpha_1} \cdots \partial y_n^{\alpha_n}$.

2. Proof of Theorem. Let (S^n, X_0) be the hypersurface. The corresponding support function φ_0 satisfies the linear elliptic partial differential equation (1.1)

$$\Delta\varphi+n\varphi=nK_{n-1}/K_n.$$

We put $\varphi_0=r+\psi_0$.
Then ψ_0 satisfies the following equation:

$$\Delta\psi+n\psi=n(K_{n-1}/K_n-r). \tag{2.1}$$

From the theory of spherical harmonics [4], the linear functions $\psi=a_1x_1+\cdots+a_{n+1}x_{n+1}$, restricted to the unit sphere, are the only solutions of the corresponding homogeneous equation $\Delta\psi+n\psi=0$. Therefore, the inhomogeneous differential equation (2.1) has solutions

$$\psi=\psi_0+a_1x_1+\cdots+a_{n+1}x_{n+1}.$$

Among those solutions there is a unique one ψ which is orthogonal to all the solutions of the homogeneous equation, namely the one with

$$a_1=\frac{-\int_{S^n}\psi_0x_1d\omega}{\int_{S^n}x_1^2d\omega},\ \cdots,\ a_{n+1}=\frac{-\int_{S^n}\psi_0x_{n+1}d\omega}{\int_{S^n}x_{n+1}^2d\omega}. \tag{2.2}$$

From the Banach's theorem and the Fredholm theory on Banach spaces [5], such unique solution ψ, by virtue of its choice, satisfies the inequality

$$\|\psi\|_{2,p}\leqslant c_1\|K_{n-1}/K_n-r\|_{0,p} \tag{2.3}$$

where c_1 is some constant depending only on p.
From Sobolev's inequalities, we have, if $p>n/2$,

$$\|\psi\|\leqslant c_2\|\psi\|_{2,p} \tag{2.4}$$

where c_2 is a constant independent of the choice of the function ψ. Therefore, we have

$$\|\psi\|\leqslant c_1c_2\|K_{n-1}/K_n-r\|_{0,p}. \tag{2.5}$$

We consider now the hypersurface (S^n, X) obtained by a translation

$$X=X_0-a,$$

where $a=(a_1,\cdots,a_{n+1})$ is the constant vector given by (2.2).
Then, the corresponding support function φ is given by

$$\varphi=r+\psi.$$

From inequality (2.5), it follows that, given an $\varepsilon>0$, if $\|K_{n-1}/K_n-r\|_{0,p}$ is sufficiently small, $\|\psi\|<\varepsilon$.
Therefore, we have

$$r-\varepsilon<\varphi<r+\varepsilon. \tag{2.6}$$

Let P_1 be the point on the hypersurface (S^n, X) at maximal distance from the origin 0 and P_2 be the point on it at minimal distance from 0. The segments $0P_1$ and $0P_2$ are perpendicular to the hypersurface at P_1, respectively P_2. Therefore, we have

$$|OP_1|=\varphi(\nu_1) \quad\text{and}\quad |OP_2|=\varphi(\nu_2).$$

From inequality (2.6), it follows that for an arbitrary point P on the hypersurface

$$r-\varepsilon<\varphi(\nu_2)=|OP_2|\leqslant|OP|\leqslant|OP_1|=\varphi(\nu_1)<r+\varepsilon.$$

Therefore, the hypersurface lies entirely within the shell between the hyperspheres of radius $r-\varepsilon$ and $r+\varepsilon$. Q.E.D.

References

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