90. The Theory of Nuclear Spaces Treated by the Method of Ranked Space. VII

By Yasujirô NAGAKURA
Science University of Tokyo

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In this paper we study a measure in the extended nuclear space, which is investigated in the papers [3]–[7].

§9. **Measure.** The nuclear space Φ following Gel'fand is constructed in a countably Hilbert space $\Phi=\bigcap_{i=1}^{\infty}\Phi_i$. From now on we shall write $\{\varphi_k\}_{n=1,2,\dots}$ in place of $\{\varphi_{k,n_1}\}_{k=1,2,\dots}$, which is an orthomormal system in the Hilbert space Φ_{n_1}.

Definition 14. Let A be a Borel set in n-dimensional space E_n generated by finite set $\{\varphi_k\}_{k=1,\dots,n}$. And we define a set Z such that

$$Z=\left\{\varphi\in\hat{\Phi},\ \sum_{i=1}^{n}(\varphi,\varphi_i)\varphi_i\in A\right\}.$$

We call it a Borel cylinder set Z with Borel base A in subspace E_n.

Thus the cylinder sets form an algebra of sets, that is,

(1) The complement of any Borel cylinder set is a Borel cylinder set.

(2) The intersection of any two Borel cylinder sets is a Borel cylinder set.

(3) The union of any two Borel cylinder sets is a Borel cylinder set.

Now, we shall extend the class of the Borel cylinder sets.

Let $\mathfrak{K}_i$ be the class of the Borel cylinder sets with Borel base in E_i.

Next, let $\mathfrak{B}_0$ be all countable unions of the elements in $\bigcup_{i=1}^{\infty}\mathfrak{K}_i$ and all complements of such unions. And we call $\mathfrak{B}_0$ Borel sets of the zeroth class. Suppose that Borel sets of class β have already been defind, where β is any ordinal number less than α such that $\alpha<\Omega$.

Then let $\mathfrak{B}_\alpha$ be all countable unions of the elements of class less than α and all complements of such unions.

Thus $\mathfrak{B}_\alpha$ is defined for all transfinite ordinal numbers less than Ω. And we call "the element of $\bigcup_{\alpha<\Omega}\mathfrak{B}_\alpha$" Borel set of Borel cylinder set.

Now, we shall define a Gaussian measure for the Borel cylinder set.

Definition 15. For the Borel cylinder set Z with Borel base A in subspace E_n, we define $\mu(Z)$ such that

$$\mu(Z)=\frac{1}{(2\pi)^{n/2}}\int_A \exp\left(\frac{-1}{2}\left[\sum_{i=1}^{n}|(\varphi,\varphi_i)|^2\right]\right)d\varphi,$$

where $d\varphi$ is Lebesgue measure with respect to the scalar product

$$(\varphi,\varphi)=\sum_{i=1}^{n}|(\varphi,\varphi_i)|^2 \qquad \text{in } E_n.$$

We call $\mu(Z)$ Gaussian measure of Z.

We shall prove that Gaussian measure defined above is countably additive.

Lemma 45. *Let $\{\alpha_k\}$ be a sequence such that $\alpha_k>0$ for all k. Then a set $S=\bigcap_{k=1}^{\infty} S_k(\alpha_k)$, where $S_k(\alpha_k)=\{\varphi\in\hat{\Phi}\,;\,|(\varphi,\varphi_k)|\leqq\alpha_k\}$ is a sequential compact set with respect to the system of semi-norms $\{|(\varphi,\varphi_k)|\}_{k=1,2,\cdots}$.*

Proof. Let $\{\varphi_\nu\}$ be a infinite subset in S. Then there exists a subsequence $\{\varphi_{1,\nu}\}$ such that the sequence $\{(\varphi_{1,\nu},\varphi_1)\}$ converges, since we have $|(\varphi_\nu,\varphi_1)|\leqq\alpha_1$. Next, there exists a subsequence $\{\varphi_{2,\nu}\}$ of $\{\varphi_{1,\nu}\}$ such that the sequence $\{(\varphi_{2,\nu},\varphi_2)\}$ converges, and so forth.

Hence the diagonal subsequence $\{\varphi_{\nu,\nu}\}$ converges with respect to the system of semi-norms $\{|(\varphi,\varphi_k)|\}_{k=1,2,\cdots}$.

Consequently, put $\varphi=\sum_{k=1}^{\infty}\beta_k\varphi_k$, where β_k is the limit of the sequence $(\varphi_{\nu\nu},\varphi_k)$, and then we have $\varphi\in S$.

Theorem 12. *Suppose that $\{Z_k\}$ is a sequence of open cylinder sets whose union is $\hat{\Phi}$. And then we have $\sum_{k=1}^{\infty}\mu(Z_k)\geqq 1$.*

Proof. For any $\varepsilon>0$, let a sequence $\{\alpha_k\}$, $\alpha_k>0$ be such that

$$\frac{1}{(2\pi)^{1/2}}\int_{|(\varphi,\varphi_k)|>\alpha_k}\exp\left(-\frac{1}{2}|(\varphi,\varphi_k)|^2\right)(d\varphi)^{(k)}<\varepsilon/2^k,$$

where $(d\varphi)^{(k)}$ is Lebesgue measure with respect to $|(\varphi,\varphi_k)|$.

And put $S=\bigcap_{k=1}^{\infty}S_k(\alpha_k)$, where $S_k(\alpha_k)=\{\varphi\in\hat{\Phi}\,;\,|(\varphi,\varphi_k)|\leqq\alpha_k\}$, then we have $S\subset\bigcup_{k=1}^{\infty}Z_k$. Any Z_k has a Borel base which is a open set in some finite dimensional subspace E_{n_k}, that is, Z_k is a open set with respect to some semi-norms.

Since the set S is a sequential compact with respect to the system of semi-norms, the set S is covered by a finite subfamily of $\{Z_k\}$, say $Z_{n1},\cdots,Z_{nh}$. Hence we have $S\subset\bigcup_{j=1}^{h}Z_{nj}$.

Let Z denote the cylinder set $\bigcup_{j=1}^{h}Z_{nj}$.

Since any Z_k has a Borel base which is a open set in some finite dimensional subspace E_{n_k}, the Z has a Borel base which is a Borel set A in finite dimensional subspace E_m such that $m=\max_{j=1\cdots h}(n_{nj})$. Then we have $P(S)\subset A$, where P is a orthogonal projection from $\hat{\Phi}$ to E_m.

Since it is clear that $P(S)=\bigcap_{k=1}^{m}P(S_k(\alpha_k))$, we have $\bigcap_{k=1}^{m}P(S_k(\alpha_k))\subset A$. If A' and $S'_k(\alpha_k)$ are the complements of A and $P(S_k(\alpha_k))$ in E_m, respectively, then we obtain $\bigcup_{k=1}^{m}S'_k(\alpha_k)\supset A'$. Let $S'_k(\alpha_k)^*$ be the cylinder set with Borel base $S'_k(\alpha_k)$ in subspace E_m. Since the cylinder set with Borel base A' in E_m is $\hat{\Phi}-Z$, we have

$$\mu\left(\bigcup_{k=1}^{m}S'_k(\alpha_k)^*\right)\geqq 1-\mu(Z).$$

The finite additivity of μ leads to

$$\sum_{k=1}^{m}\mu(S'_k(\alpha_k)^*)\geqq 1-\mu(Z)=1-\mu\left(\bigcup_{j=1}^{h}Z_{nj}\right)\geqq 1-\sum_{j=1}^{h}\mu(Z_{nj}).$$

Hence we have

$$\sum_{k=1}^{m} \mu(S'_k(\alpha_k)^*) \geqq 1 - \sum_{j=1}^{h} \mu(Z_{nj}).$$

By the hypothesis we see

$$\mu(S'_k(\alpha_k)^*) = \frac{1}{(2\pi)^{1/2}} \int_{|(\varphi,\varphi_k)|>\alpha_k} \exp\left(-\frac{1}{2}|(\varphi,\varphi_k)|^2\right) (d\varphi)^{(k)} < \varepsilon/2^k.$$

Then we have

$$\sum_{k=1}^{m} \varepsilon/2^k \geqq 1 - \sum_{j=1}^{h} \mu(Z_{nj}),$$

hence

$$\varepsilon > 1 - \sum_{j=1}^{h} \mu(Z_{nj}),$$

and therefore

$$\sum_{j=1}^{\infty} \mu(Z_{nj}) > 1 - \varepsilon.$$

Since ε is arbitrary, it follows from this that

$$\sum_{k=1}^{\infty} \mu(Z_k) \geqq 1,$$

which completes the proof.

Theorem 13. *In order that Gaussian measure μ be countably additive, it is necessary and sufficient that $\sum_{k=1}^{\infty} \mu(Z_k) = 1$ for any family of nonintersecting Borel cylinder sets $\{Z_k\}$ such that $\hat{\Phi} = \bigcup_{k=1}^{\infty} Z_k$.*

Proof. We shall prove that this condition is sufficient. Now, suppose that Z is some cylinder set and $Z = \bigcup_{k=1}^{\infty} Z_k$ is a decomposition into nonintersecting cylinder sets Z_k. Then we obtain

$$\hat{\Phi} = (\hat{\Phi} - Z) \cup \left(\bigcup_{k=1}^{\infty} Z_k\right).$$

By the hypothesis, we obtain

$$\mu(\hat{\Phi} - Z) + \sum_{k=1}^{\infty} \mu(Z_k) = 1.$$

From the finite additivity of μ, we have

$$\mu(\hat{\Phi} - Z) = 1 - \mu\left(\bigcup_{k=1}^{\infty} Z_k\right).$$

Consequently we obtain

$$\mu\left(\bigcup_{k=1}^{\infty} Z_k\right) = \sum_{k=1}^{\infty} \mu(Z_k).$$

Theorem 14. *In order that Gaussian measure μ be countably additive, it is necessary and sufficient that $\sum_{k=1}^{\infty} \mu(Z_k) \geqq 1$ for any family of Borel cylinder sets $\{Z_k\}$ such that $\hat{\Phi} = \bigcup_{k=1}^{\infty} Z_k$.*

Proof. We shall prove that this condition is sufficient. Suppose that the Borel cylinder sets $\{Z_k\}$ whose union is $\hat{\Phi}$ are nonintersecting.

Then the finite additivity of μ leads to

$$\sum_{k=1}^{\infty} \mu(Z_k) \leqq 1$$

By the hypothesis, we obtain $\sum_{k=1}^{\infty} \mu(Z_k) = 1$, then μ is countably additive by Theorem 13.

Theorem 15. *If X is a subset in $\hat{\Phi}$ such that $\mu(X)=0$, we have $\mu(X+\varphi_k)=0$ for any element φ_k of the orthonormal system $\{\varphi_k\}$ in the Hilbert space Φ_{n_1}.*

Proof. Let X_n be the projection of X into the subspace generated by the element φ_n. Then we have

$$\mu(X)=\prod_{i=1}^{\infty}\left(\frac{1}{(2\pi)^{1/2}}\int_{X_i}\exp\left(-\frac{1}{2}|(\varphi,\varphi_i)|^2\right)(d\varphi)^{(i)}\right),$$

where $(d\varphi)^{(i)}$ is Lebesgue measure with respect to $|(\varphi,\varphi_i)|$.

In the other hand, we see

$$\begin{aligned}\mu(X+\varphi_k)=&\prod_{i=1}^{k-1}\left(\frac{1}{(2\pi)^{1/2}}\int_{X_i}\exp\left(-\frac{1}{2}|(\varphi,\varphi_i)|^2\right)(d\varphi)^{(i)}\right)\\&\times\left(\frac{1}{(2\pi)^{1/2}}\int_{X_k+\varphi_k}\exp\left(-\frac{1}{2}|(\varphi,\varphi_k)|^2\right)(d\varphi)^{(k)}\right)\\&\times\prod_{i=k+1}^{\infty}\left(\frac{1}{(2\pi)^{1/2}}\int_{X_i}\exp\left(-\frac{1}{2}|(\varphi,\varphi_i)|^2\right)(d\varphi)^{(i)}\right).\end{aligned}$$

Hence, if we assume that $\mu(X+\varphi_k)\neq 0$, we obtain

$$\frac{1}{(2\pi)^{1/2}}\int_{X_k+\varphi_k}\exp\left(-\frac{1}{2}|(\varphi,\varphi_k)|^2\right)(d\varphi)^{(k)}\neq 0.$$

Consequently, we have

$$\frac{1}{(2\pi)^{1/2}}\int_{X_k}\exp\left(-\frac{1}{2}|(\varphi,\varphi_k)|^2\right)(d\varphi)^{(k)}\neq 0.$$

Then we see $\mu(X)\neq 0$. Q.E.D.

The investigation in this paper and in [3]–[7] of the References may be considered as an answer to the 10-th of the problems given by K. Kunugi at the end of his "On the method of ranked spaces (in Japanese)", Noda Mathematical Pamphlet Series 1 (1969) published by The Seminar of Ranked Space. I wish to thank Prof. Kinjirô Kunugi and Dr. Kazuhisa Shima for their valuable discussions and suggestions.

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