86. On Remainder Estimates in the Asymptotic Formula of the Distribution of Eigenvalues of Elliptic Operators

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1. Introduction. Let Ω be a domain in R^n with boundary uniformly regular of class $m+1$. Let $\mathcal{A}=\sum a_\alpha(x)D^\alpha$ be a formally self-adjoint positively elliptic operator of order m with coefficients defined and bounded in Ω. Let A be a self-adjoint realization of $\mathcal{A}$ with domain contained in $W_2^m(\Omega)$. By $N(t)$ we denote the number of eigenvalues $\leqq t$ of A. Assuming that the highest order coefficients of A are continuous R. Beals [2] investigated the asymptotic behaviour of the resolvent kernel and spectral function of A, and as an application of his results he proved that the asymptotic formula

$$N(t)=c_0t^{n/m}+O(t^{(n-\theta)/m}),\qquad t\to\infty \tag{1.1}$$

holds for any $0<\theta<h/(h+3)$ provided that the top-order coefficients of $\mathcal{A}$ are uniformly Hölder continuous of order h. The object of this note is to improve the remainder estimate in (1.1) and prove the following theorem.

Theorem. *Suppose Ω is bounded. Let A be a self-adjoint semi-bounded realization of $\mathcal{A}$ with domain contained in $W_2^m(\Omega)$. If $m\leqq n/2$ we make the additional assumption that A satisfies the resolvent condition for $2\leqq q\leqq n/m+\varepsilon$ with some $\varepsilon>0$ ([2]), i.e. for each $\delta>0$ there are constants c_1 and c_2 such that $(A-\lambda)^{-1}$ induces a bounded operator from $L^q(\Omega)$ to $W_q^m(\Omega)$ and*

$$\|(A-\lambda)^{-1}u\|_q\leqq c_1|\lambda|^{-1}\|u\|_q$$

for all $u\in L^q(\Omega)$, $|\lambda|\geqq c_2$, $|\arg\lambda|\geqq\delta$. If the highest order coefficients of $\mathcal{A}$ are uniformly continuous of order h, then

$$N(t)=c_0t^{n/m}+O(t^{(n-\theta)/m}) \tag{1.2}$$

for any $0<\theta<h/(h+2)$, where

$$c_0=(2\pi)^{-n}\int_\Omega\int_{a(x,\xi)<1}d\xi dx.$$

If the highest order coefficients of $\mathcal{A}$ belong to the class C^{1+h} in some domain containing $\bar\Omega$, then (1.2) holds for any $0<\theta<(h+1)/(h+3)$.

2. Outline of the proof of the main theorem.

If $m>n/2$, we have only to apply the main theorem of K. Maruo [3] to the sesquilinear form (Au,Av). Hence, in what follows we assume that $m\leqq n/2$.

Lemma 1 (R. Beals [2]). *If S and T are bounded operators in $L^2(\Omega)$ such that the ranges of S and T^* are contained in $L^\infty(\Omega)$. Then* *the operator ST has a bounded kernel k satisfying*

$$|k(x, y)| \leqq \|S\|_{L^2 \to L^\infty} \|T^*\|_{L^2 \to L^\infty}.$$

For a complex number μ we denote by $d(\mu)$ the distance from μ to the positive real axis, and for $x \in \Omega$ we write $\delta(x) = \min\{\text{dist}(x, \partial\Omega), 1\}$. Let A_D be the restriction of $\mathcal{A}$ to $\mathring{H}_{m/2}(\Omega) \cap H_m(\Omega)$. Let p be an integer such that $[p/2] > n/2m$. We denote by $K_\mu(x, y)$ and $K_\mu^D(x, y)$ the kernels of $(A^p - \mu)^{-1}$ and $(A_D^p - \mu)^{-1}$ respectively.

Lemma 2. *For any $x \in \Omega$, $l > 0$, and a complex number μ not lying on the positive real axis*

$$|K_\mu(x, x) - K_\mu^D(x, x)| \leqq C \frac{|\mu|^{n/pm}}{d(\mu)} \left(\frac{|\mu|^{1-1/pm}}{\delta(x) d(\mu)} \right)^l \tag{1.1}$$

where C is a constant dependent on l but not on x or μ.

Proof. Let $p = s + t$, where s and t are integers $> n/2m$. Following Beals [2] we choose $q_1, q_2, \cdots, q_{s+1}$ such that

$$0 = q_{s+1}^{-1} < q_s^{-1} < m/n < q_{s-1}^{-1} < \cdots < q_2^{-1} < q_1^{-1} = 1/2$$

and $q_j^{-1} - q_{j+1}^{-1} < m/n$ for $j = 1, \cdots, s$, and $r_1, \cdots, r_{t+1}$ similarly. Let $\lambda_1, \cdots, \lambda_p$ be the roots of $\lambda^p = \mu$. Exactly one of them lies in the sector $-\pi/p < \arg \lambda \leqq \pi/p$, and we take this to be λ_s. Then

$$(A^p - \mu)^{-1} = \prod_{j \leq s} S_j \prod_{j > s} S_j \tag{1.2}$$

$$(A_D^p - \mu)^{-1} = \prod_{j \leq s} T_j \prod_{j > s} T_j, \tag{1.3}$$

where $S_j = (A - \lambda_j)^{-1}$ and $T_j = (A_D - \lambda_j)^{-1}$. Let x_0 be a fixed point of Ω and $\zeta_1, \cdots, \zeta_p$ be functions in $C_0^\infty(\Omega)$ such that $\zeta_j(x_0) = 1$ and $|\nabla^k \zeta_j(x)| \leqq \text{const.}\ \delta(x_0)^{-k}$ for $j = 1, \cdots, p$ and $k = 0, 1, \cdots, m-1$. By elementary calculation

$$\prod_{j \leq s} \zeta_j \prod_{j \leq s} S_j = \prod_{j \leq s} \zeta_j S_j + R_1, \tag{1.4}$$

$$\prod_{j > s} S_j \prod_{j > s} \zeta_j = \prod_{j > s} S_j \zeta_j + R_2 \tag{1.5}$$

where R_1 (resp. R_2) is a sum of terms having

$$\zeta_k S_i - S_i \zeta_k = S_i [A, \zeta_k] S_i$$

as a factor for some $k \leq s$ (resp. $k > s$). Combining (1.2), (1.4), (1.5) we get

$$\begin{aligned} &\prod_{j \leq s} \zeta_j (A^p - \mu)^{-1} \prod_{j > s} \zeta_j \\ &= \prod_{j \leq s} \zeta_j S_j \prod_{j > s} S_j \zeta_j + \prod_{j \leq s} \zeta_j S_j \cdot R_2 + R_1 \prod_{j > s} S_j \prod_{j > s} \zeta_j, \end{aligned} \tag{1.6}$$

and similarly for $\prod_{j \leq s} \zeta_j (A_D^p - \mu)^{-1} \prod_{j > s} \zeta_j$. Hence

$$\begin{aligned} &\zeta_1 \cdots \zeta_s \{(A^p - \mu)^{-1} - (A_0^p - \mu)^{-1}\} \zeta_{s+1} \cdots \zeta_p \\ &= \left\{ \prod_{j \leq s} \zeta_j S_j \prod_{j > s} S_j \zeta_j - \prod_{j \leq s} \zeta_j T_j \prod_{j > s} T_j \zeta_j \right\} + \cdots. \end{aligned} \tag{1.7}$$

We estimate the kernels of all terms on the right side of (1.7). Rewriting the inside of the bracket and noting that

$$\zeta_k (S_k - T_k) = T_k [\mathcal{A}, \zeta_k] S_k - T_k [\mathcal{A}, \zeta_k] T_k$$

we investigate only the kernel of

$$\zeta_1 T_1 \cdots \zeta_{k-1} T_{k-1} T_k [\mathcal{A}, \zeta_k] S_k \zeta_{k+1} S_{k+1} \cdots \zeta_s S_s \prod_{j>s} S_j \zeta_j \tag{1.8}$$

for $k \leq s$ since the remaining ones in the bracket can be dealt with in a similar manner. By Beals [2] we have

$$\|S_s u\|_{q_2} \leqq c|\mu|^{(n/pm)(1/2-1/q_2)-1}|\mu|/d(\mu)\|u\|_2 \tag{1.9}$$

$$\|S_{s-j+1} u\|_{q_{j+1}} \leqq c|\mu|^{(n/pm)(1/q_j-1/q_{j+1})-1}\|u\|_{q_j}, \qquad 2 \leq j \leq s, \tag{1.10}$$

$$\|S^*_{p-j+1} u\|_{r_j} \leqq c|\mu|^{(n/pm)(1/r_{j+1}-1/r_j)-1}\|u\|_{r_{j+1}}, \qquad 1 \leq j \leq t. \tag{1.11}$$

With the aid of Lemma 1 and (1.9), (1.10), (1.11), it is shown that the kernel of (1.8) is bounded by $C(|\mu|^{n/pm}/d(\mu))(|\mu|^{1-1/pm}/\delta(x_0)d(\mu))$. It is not difficult to show that the kernels of the remaining terms on the right side of (1.7) is dominated by the same value. Hence (1.1) is established for $l=1$. For general integers $l \geqq 1$ (1.3) can be proved by induction considering

$$\eta_1 \cdots \eta_{l-1} \zeta_1 \cdots \zeta_s \{(A^p - \mu)^{-1} - (A_D^p - \mu)^{-1}\} \zeta_{s+1} \cdots \zeta_p \eta_1 \cdots \eta_{l-1}$$

where $\eta_1, \cdots, \eta_{l-1}$ are functions in $C_0^\infty(\Omega)$ such that $\eta_j(x_0)=1, \eta_j \eta_{j+1} = \eta_j$, $j=1, \cdots, l-1$, and $\eta_l \zeta_1 \cdots \zeta_p = \zeta_1 \cdots \zeta_p$. It is clear that (1.3) is valid for $l=0$. That (1.3) holds for nonintegral values of l follows by induction.

Lemma 2 shows that we have only to consider the case of the Dirichlet boundary conditions, and hence in what follows we assume $A = A_D$. Let x_0 be a fixed point of Ω and let A_0 be the restriction of $\sum_{|\alpha|=m} a_\alpha(x_0) D^\alpha$ to $\mathring{H}_{m/2}(\Omega) \cap H_m(\Omega)$. By $K_\mu^0(x, y)$ we denote the resolvent kernel of A_0^p. For $\eta \in C_0^\infty(R^n)$ the range of $\eta(A-\lambda)^{-1}$ or $\eta(A_0-\lambda)^{-1}$ is contained in $D(A)=D(A_0)$ whether the support of η is contained in Ω or not since we are confining ourselves to the Dirichlet boundary conditions. Taking this remark into consideration we may prove the following lemma by essentially the same method as Lemma 2.

Lemma 3. *If the highest order coefficients of $\mathcal{A}$ are uniformly Hölder continuous of order h, then for*

$$j=1, 2, \cdots, \varepsilon > 0, |\mu|^{1-1/pm}/\varepsilon d(\mu) \leqq 1,$$

$$|K_\mu(x, x) - K_\mu^0(x, x)| \leqq C|\mu|^{n/pm-1}\left(\frac{|\mu|}{d(\mu)}\right)^2 \left\{\varepsilon^h + \left(\frac{|\mu|^{1-1/pm}}{\varepsilon d(\mu)}\right)^j\right\}$$

where C is a constant depending on j but not on x, ε, μ.

Now we invoke Theorem 3.2 of S. Agmon [1]. Using this theorem applied to A_0 and Lemma 3 and following the argument of K. Maruo [3] we obtain the first half of our main theorem. The remaining part of the theorem can be proved in a similar manner again following the argument of [3].

References

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[2] R. Beals: Asymptotic behavior of the Green's function and spectral function of an elliptic operator. J. Func. Anal., **5**, 484–503 (1970).

[3] K. Maruo: Asymptotic distribution of eigenvalues of non-symmetric operators associated with strongly elliptic sesquilinear forms (to appear).