104. *A Pointwise Ergodic Theorem for Positive Bounded Operator*

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1. Introduction and the theorem. The purpose of this note is to prove a pointwise ergodic theorem for a positive bounded linear operator which generalizes those induced by non-singular measurable transformations and Markov processes with an invariant measure. Throughout this note, let $(X, \mathfrak{B}, m)$ be a finite measure space. We denote the norm and the operator norm in $L_p(X)$ by $\| \ \|_p$ $(1 \leqslant p \leqslant \infty)$. Let T be a positive bounded linear operator defined on $L_1(X)$. (The positivity means that $Tf \geqslant 0$, if $f \geqslant 0$.) Put $S_n = \sum_{k=0}^{n-1} T^k$, where $T^0 = I$ (identity). In the sequel we assume that the operator T satisfies the following conditions.

(A) There exists a constant $K > 0$ such that
$$\|(1/n)S_n\|_1 \leqslant K \quad \text{and} \quad \|(1/n)S_n\|_\infty \leqslant K (n=1,2,\cdots),$$

(B) $\lim_{n\to\infty} \|(T^n/n)f\|_1 = 0$ for any $f \in L_1(X)$ and $\lim_{n\to\infty} \|(T^n/n)f\|_\infty = 0$ for any $f \in L_\infty(X)$,

(C) If $f \geqslant 0, f \in L_1(X)$ and $\liminf_{n\to\infty} \|(S_n/n)f\|_1 = 0$, then $f = 0$.

We shall prove the following

Theorem. *Let T be a positive bounded linear operator on $L_1(X)$. If the operator T satisfies three conditions* (A), (B) *and* (C), *then a pointwise ergodic theorem holds for T, that is, the limit*
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (T^k f)(x)$$
exists almost everywhere for any $f \in L_1(X)$ and it is in $L_1(X)$.

Remark. The operator in the theorem includes those induced by measure preserving transformations (the Birkhoff's pointwise ergodic theorem). Consider an operator induced by a non-singular measurable transformation. Then we have a pointwise ergodic theorem for the operator only if the operator satisfies the above condition (C). For the operator induced by a Markov process, there exists a finite invariant measure μ with $\mu \sim m$ if and only if the operator satisfies the above condition (C) [3]. The operator in the theorem includes a positive invertible operator T with $\sup_{-\infty<n<\infty} \|T^n\|_1 < \infty$ and $\sup_{-\infty<n<\infty} \|T^n\|_\infty < \infty$.

2. The proof of the theorem.

We have the mean ergodic theorem for T.

Lemma 1. *The limit*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}(T^k f)(x)$$

exists strongly for any $f\in L_1(X)$ and almost everywhere for any f in the set A which is dense in $L_1(X)$, where $A=\{h: h=f+Tg-g, Tf=f, g$ is essentially bounded, $f,g\in L_1(X)\}$.

Proof. Since a sequence $((S_n/n)f)(n=1,2,\cdots)$ is weakly sequentially compact for any $f\in L_1(X)$ by (A), the first part of the lemma is obvious from Yosida-Kakutani-Riesz theorem [2, 4, 5]. The second part of the lemma follows from (A), (B) and the proof of the Yosida-Kakutani-Riesz theorem.

Lemma 2. $\limsup_{n\to\infty}((S_n/n)f)(x)<\infty$ *a.e. for any* $f\in L_1(X)$.

Proof. We can assume $f\geqslant 0$. Put $E=\{x: \limsup_{n\to\infty}((S_n/n)f)(x)=\infty\}$ and $E(a)=\{x: \limsup_{n\to\infty}(S_n/n)(f(x)-a)>0\}$, where a is an arbitrary positive number. We use the Chacon-Ornstein lemma.

Lemma (Chacon-Ornstein) [1]. *If* $\sup_{n\geqslant 1}(S_n f)(x)>0$ *on a set E, then there exist sequences $\{d_k\}$ and $\{f_k\}$ of non-negative functions such that*

$$(1)\qquad \sum_{k=0}^{\infty}d_k=f^- \quad \text{on} \quad E$$

and

$$(2)\qquad T^j f^+=\sum_{k=0}^{j}T^{j-k}d_k+f_j(0\leqslant j).$$

Remark. Though the lemma was proved under an assumption with $\|T\|_1\leqslant 1$, their proof of (1) and (2) is obtained without appealing this assumption.

Since $E\subset E(a)=\{x: \sup_{n\geqslant 1}S_n(f(x)-a)>0\}$ by (A), we can apply the lemma for E and $f-a$ and get sequences $\{d_{k,a}\}$ and $\{f_{k,a}\}$ of non-negative functions such that

$$(3)\qquad \sum_{k=0}^{\infty}d_{k,a}=(f-a)^- \quad \text{on} \quad E$$

and

$$(4)\qquad T^j(f-a)^+=\sum_{k=0}^{j}T^{j-k}d_{k,a}+f_{j,a}.$$

Since S_n/n is a positive operator and $f_{j,a}\geqslant 0$, we have by (4)

$$\int\frac{S_n}{n}T^j(f-a)^+dm\geqslant\int\frac{S_n}{n}\left(\sum_{k=0}^{j}T^{j-k}d_{k,a}\right)dm.$$

By Lemma 1 and (B) we have $s\text{-}\lim_{n\to\infty}(S_n/n)(T^j g-g)=0$ for any $g\in L_1(X)$ and therefore

$$\int s\text{-}\lim_{n\to\infty}\frac{S_n}{n}(f-a)^+dm\geqslant\int s\text{-}\lim_{n\to\infty}\frac{S_n}{n}\sum_{k=0}^{j}d_{k,a}dm.$$

By (A),

$$K\int(f-a)^+\,dm \geqslant \int s\text{-}\lim_{n\to\infty}\frac{S_n}{n}\sum_{k=0}^{j} d_{k,a}dm.$$

Since a sequence $(s\text{-}\lim_{n\to\infty}(S_n/n)\sum_{k=0}^{j} d_{k,a})(j=0,1,2,\cdots)$ of non-negative functions is increasing, by the Fatou lemma we have

$$K\int(f-a)^+\,dm \geqslant \int s\text{-}\lim_{n\to\infty}\frac{S_n}{n}\sum_{k=0}^{\infty} d_{k,a}dm. \tag{5}$$

Let ε be an arbitrary positive number. If a is large enough, then it follows from (3) that there exists a measurable set $F(F\subset E)$ such that $m(E-F)<\varepsilon$ and $\sum_{k=0}^{\infty} d_{k,a}\geqslant\chi_F$, where χ_F is the characteristic function of F. If a tends to infinity we have by (5) and the positivity of $s\text{-}\lim_{n\to\infty}(S_n/n)$,

$$0=\lim_{a\to\infty}K\int(f-a)^+\,dm \geqslant \int s\text{-}\lim_{n\to\infty}\frac{S_n}{n}\chi_F dm.$$

By (C) we have $m(F)=0$. Since ε is arbitrary we have $m(E)=0$.

The proof of the theorem (Cf. K. Yosida [4, 5]). The proof is obtained by Lemma 1, Lemma 2 and the Banach convergence lemma.

Lemma [2, 4, 5]. *Let (T_n) $(n=1,2,\cdots)$ be a sequence of bounded linear operators from a Banach space L into the Fréchet space (S). If $\limsup_{n\to\infty}|(T_nf)(x)|<\infty$ for any $f\in L$, then the (not necessarily linear) operator $\tilde{T}$ defined by*

$$(\tilde{T}f)(x)=\limsup_{n\to\infty}(T_nf)(x)-\liminf_{n\to\infty}(T_nf)(x)$$

is continuous as an operator defined on L into (S). (The quasi-norm of (S) is defined by

$$\|f\|=\int\frac{|f(x)|}{1+|f(x)|}dm(x)$$

for any measurable function $f\in(S)$.

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