132. On Characters and Unipotent Elements of Finite Chevelley Groups

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The purpose of the present paper is to give some results concerning (complex) characters and unipotent elements of finite Chevelley groups. Main results are proved by two simple lemmas stated in section 1. Throughout the paper G denotes a connected reductive linear algebraic group defined over a finite field k of q elements. For simplicity we also assume that G has a maximal torus T which splits over k. If L is an algebraic subgroup of G defined over k, $L(k)$ denotes the finite group of its k-rational elements. If S is a finite set, $|S|$ denotes the number of its elements. For a finite group H and class functions ϕ_1 and ϕ_2 on H, the inner product $(\phi_1, \phi_2)_H$ is defined by

$$(\phi_1, \phi_2)_H = \sum_{x \in H} \phi_1(x)\overline{\phi_2(x)}/|H|.$$

If K is a subgroup of H and θ is a class function on K, $i[\theta\,; K \to H]$ (or $i[\theta]$) denotes the class function on H induced by θ.

1. Let W be the Weyl group of G relative to T and B a fixed Borel k-subgroup of G containing T. B determines a set Φ_+ of positive roots and a set Δ of simple roots in the system Φ of roots of G relative to T. For each subset δ of Δ, let P_δ be the parabolic k-subgroup corresponding to δ and G_δ, U_δ its Levi k-subgroup and unipotent radical (see § 3 of the paper of A. Borel and J. Tits in Publ. de Math. I. H. E. S. n°27 (1965)). G_δ is connected reductive and the root system Φ_δ of G_δ relative to T is spanned by δ. We denote by W_δ the Weyl group of G_δ relative to T.

Lemma 1 (L. Solomon, C. W. Curtis). (a) *Let 1_δ be the 1-character of W_δ and ε the alternating character of W. Then*

$$\varepsilon = \sum_{\delta \subset \Delta} (-1)^{|\delta|}\, i[1_\delta\,; W_\delta \to W].$$

(b) *Let $P_\delta^1(k)$ be the set of unipotent elements of $P_\delta(k)$ and θ_δ be the class function on $P_\delta(k)$ defined by*

$$\theta_\delta(x) = \begin{cases} 1 & \text{if } x \in P_\delta^1(k), \\ 0 & \text{otherwise.} \end{cases}$$

If we put

$$\Theta = \sum_{\delta \subset \Delta} (-1)^{|\delta|}\, i[\theta_\delta\,; P_\delta(k) \to G(k)], \tag{1.1}$$

then

$$\Theta(x) = \begin{cases} q^m & \text{if } x = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $m=|\Phi_+|$.

These are proved in [2] and [5].

In the next lemma, we must take for G the adjoint group extended by the diagonal automorphisms (G' in [8, p. 263]). Let U be the maximal unipotent subgroup of G contained in B so that $B=TU$. For $a\in\Phi$, $X_a=\{x_a(t)\}$ denotes the one-parameter unipotent subgroup defined by a. Let ψ_δ ($\delta\subset\Delta$) be the following linear character of $U(k)$:

$$\psi_\delta(x_a(t))=\begin{cases}\alpha(t) \quad (t\in k) & \text{if } a\in\delta,\\ 1 & \text{if } a\in\Phi_+\backslash\delta,\end{cases}$$

where α is a fixed nontrivial character of k.

Lemma 2. *Let G be as explained above. Define a function Ψ on $G(k)$ by*

$$\Psi=\sum_{\delta\subset\Delta}(-1)^{|\delta|}\,i[\psi_\delta\,;\,U(k)\to G(k)]. \tag{1.2}$$

Then

$$\Psi(x)=\begin{cases}q^{|\Delta|} & \text{if } x \text{ is a regular unipotent element in } G(k),\\ 0 & \text{otherwise.}\end{cases}$$

Proof. It is sufficient to prove this for $x\in U(k)$. Then x can be expressed uniquely as $x=\prod x_a(t_a)$ ($a\in\Phi_+$) with the product taken in any fixed order. Hence

$$\begin{aligned}\sum_{\delta\subset\Delta}(-1)^{|\delta|}\psi_\delta(x)&=\sum_{\delta\subset\Delta}(-1)^{|\delta|}\psi_\delta\left(\prod x_a(t_a)\right)\\ &=\sum_{\delta\subset\Delta}(-1)^{|\delta|}\prod_{a\in\delta}\alpha(t_a)=\prod_{a\in\Delta}(1-\alpha(t_a)).\end{aligned} \tag{1.3}$$

Since $x=\prod x_a(t_a)$ is regular unipotent if and only if $t_a\neq 0$ for all $a\in\Delta$ ([1; p. 220]), we see by (1.3) that $\psi(x)=0$ if x is not regular unipotent. Hence $\Psi(x)=i[\psi](x)=0$ also. Now assume that x is not regular unipotent. In this case $g^{-1}xg\in U(k)$ for $g\in G(k)$ if and only if $g\in B(k)=T(k)U(k)$ (see [1; p. 220]). If $g=tu$ ($t\in T(k), u\in U(k)$), $\psi_\delta(g^{-1}xg)=\prod_{a\in\delta}\alpha(a(t)t_a)$. Hence

$$\begin{aligned}i[\psi_\delta\,;\,U(k)\to G(k)](x)&=\sum_{g\in B(k)}\psi_\delta(g^{-1}xg)/|U(k)|\\ &=\sum_{t\in T(k)}\prod_{a\in\delta}\alpha(a(t)t_a)=(-1)^{|\delta|}\,|\{t\in T(k)\,|\,a(t)=1 \quad \text{for } a\in\delta\}|\end{aligned}$$

by the definition of G and the fact: $\sum_{t\in k^\times}\alpha(t)=-1$. Thus we have

$$\Psi(x)=\sum_{\delta\subset\Delta}|\{t\in T(k)\,|\,a(t)=1 \quad \text{for } a\in\delta\}|,$$

which equals to $q^{|\Delta|}$ (see [8; p. 263]) as required.

2. As a first application of Lemma 1 we give a simple proof of the following theorem. The original proof can be found in [7; § 14, § 15].

Theorem 1 (R. Steinberg). (a) *The number of maximal tori of G defined over k is q^{2m}.* (b) *The number of unipotent elements of G defined over k, i.e. of $G(k)$ is also q^{2m}.*

Proof. Our method is similar to the one used in [5] to prove a multiplicative formula for the orders of the finite Chevelley groups. Consider W as a finite reflection group which acts on $\hat{T}\otimes_{\mathbf{Z}}\mathbf{R}$ ($\hat{T}$ dual of T). Then by [6; (5)] we have

(2.1) $$\frac{1}{|W|}\sum_{w\in W}\frac{\det w}{\det(w-t)}=\frac{1}{(1-t^{m_1+1})\cdots(1-t^{m_n+1})}$$

and

(2.2) $$\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det(w-t)}=\frac{t^m}{(1-t^{m_1+1})\cdots(1-t^{m_n+1})},$$

where $m_1,\cdots,m_n$ are the exponents for W. (Recall that $m=m_1+\cdots+m_n$ ([8; p. 136]).) Let $m_1(\delta),\cdots,m_n(\delta)$ be the exponents for W_δ. Then by (2.2) we have

(2.3) $$\frac{1}{|W_\delta|}\sum_{w\in W_\delta}\frac{1}{\det(w-t)}=\frac{t^{m(\delta)}}{(1-t^{m_1(\delta)+1})\cdots(1-t^{m_n(\delta)+1})},$$

where $m(\delta)=m_1(\delta)+\cdots+m_n(\delta)$. Consider the following sum

(2.4) $$\sum_{\delta\subset\Delta}(-1)^{|\delta|}\left\{\frac{1}{|W_\delta|}\sum_{w\in W_\delta}\frac{1}{\det(w-t)}\right\}.$$

It is easy to see that this equals to

$$\frac{1}{|W|}\sum_{w_i}\frac{1}{\det(w_i-t)}\{\sum_{\delta\subset\Delta}(-1)^{|\delta|}i[1_\delta\,;\,W_\delta\to W](w_i)\},$$

where the first sum is over a set of representatives of the conjugacy classes of W. Hence by Lemma 1 (a) and the fact that $\det w=\varepsilon(w)$ $(w\in W)$, (2.4) equals to

$$\frac{1}{|W|}\sum_{w\in W}\frac{\det w}{\det(w-t)}.$$

This combined with (2.1) and (2.3) leads to

(2.5) $$\sum_{\delta\subset\Delta}(-1)^{|\delta|}\frac{t^{m(\delta)}}{(1-t^{m_1(\delta)+1})\cdots(1-t^{m_n(\delta)+1})}=\frac{1}{(1-t^{m_1+1})\cdots(1-t^{m_n+1})}.$$

Let G be the universal Chevelley group in the sense of [8]. There is no loss of generality in assuming this. Then by (2.5) and the multiplicative formula for the orders of the finite Chevelley groups ([5], [8; § 9]),

(2.6) $$\sum_{\delta\neq\Delta}(-1)^{|\delta|}\frac{q^{2m(\delta)}}{|G_\delta(k)|}=\frac{q^m-(-1)^{|\Delta|}q^{2m}}{|G(k)|},$$

where $G_\delta(k)$ is the group defined in 1. Let $G^1(k)$ be the set of all unipotents of $G(k)$. Consider the sum:

$$\sum_{u\in G^1(k)}\Theta(u).$$

By Lemma 1 (b) and the Froberius reciprocity theorem we have

(2.7) $$\sum_{\delta\neq\Delta}(-1)^{|\delta|}\frac{|P_\delta^1(k)|}{|P_\delta(k)|}=\frac{q^m-(-1)^{|\delta|}|G^1(k)|}{|G(k)|}.$$

Using the semidirect decomposition $P_\delta(k)=G_\delta(k)U_\delta(k)$ we see that $P_\delta^1(k)=G_\delta^1(k)U_\delta(k)$. Hence (2.7) equals to

(2.8) $$\sum_{\delta\neq\Delta}(-1)^{|\delta|}\frac{|G_\delta^1(k)|}{|G_\delta(k)|}=\frac{q^m-(-1)^{|\delta|}|G^1(k)|}{|G(k)|}.$$

We can now prove Theorem 1 (b) by the induction on $|\Delta|$. If $|\Delta|=0$, this is trivial. For $\delta\subsetneq\Delta$ $|G_\delta^1(k)|=q^{2m(\delta)}$ by the induction hypothesis.

Thus by (2.6) and (2.8) we get the required result $|G^1(k)|=q^{2m}$. As for the part (a) of Theorem 1, (2.2) is essential for its proof. See the Steinberg's paper [7; § 14] for the details.

3. In this section, we announce a theorem which is a generalization of the Green's map used in the character theory of $GL_n(k)$ (see [3] and [1; Part D]). This can be proved by a technique similar to the one used in the previous section. The detailed proof is given in the author's forthcoming paper [4]. Let P be a parabolic k-subgroup of G and $B(P(k))$ the set of all class functions on $P(k)$ which satisfy the condition

(3.1) $$\phi(x)=\phi(x_s) \qquad (x \in P(k)),$$

where x_s is the semisimple part of x. Let $T_1, T_2, \cdots, T_l$ be a set of representatives of the $G(k)$-conjugacy classes of maximal tori of G defined over k. For $\phi \in B(k)$, define the function $\phi^P_{T_i}$ on $T_i(k)$ by

(3.2) $$\phi^P_{T_i}(t)=\sum\nolimits_{Q\supset T_i} \phi^Q(t) \qquad (t \in T_i(k)),$$

where the sum is over the set of all parabolic k-subgroups Q containing T_i which are $G(k)$-conjugate to P and

$$\phi^Q(t)=\phi(gtg^{-1})$$

if $Q=g^{-1}Pg$ $(g \in G(k))$. Let W_i be the finite group $N_{G(k)}(T_i)/T_i(k)$. Then $\phi^P_{T_i}$ is invariant under W_i.

Theorem 2. *Let P, P' be parabolic k-subgroups of $G(k)$ and ϕ_1, ϕ_2 elements of $B(P(k))$ and $B(P'(k))$ respectively. Then*

$$(\phi_1, \phi_2)_{G(k)}=\sum\nolimits_{i=1}^{l} |w_i|^{-1}(\phi^P_{1T_i}, \phi^{P'}_{2T_i})_{T_i(k)},$$

where $\phi^P_{T_j}$'s are defined by (3.2).

In the special case $G=GL_n$, one can reformulate this as follows.

Theorem 3. *Let $G=GL_n$. There exists a linear map $\phi \to \phi^{\wedge i}$ from the complex vector space of class functions on $G(k)$ onto the one of W_i-invariant functions on $T_i(k)$ which satisfies*

$$(\phi_1, \phi_2)_{G(k)}=\sum\nolimits_{i=1}^{l} |W_i|^{-1} (\phi_1^{\wedge i}, \phi_2^{\wedge i})_{T_i(k)}.$$

This formula was originally proved by J. A. Green [3] using a combinatorial method.

4. Proposition 1. *Let χ be a character of $G(k)$ which is a cusp form* (see [1; Part C]). *Then*

(4.1) $$\sum\nolimits_{u \in G^1(k)} \chi(u)=(-1)^{|\Delta|}\chi(1)q^m \qquad (m=|\Phi_+|).$$

Proposition 2. *Let G be as in Lemma 2 and χ an irreducible character of $G(k)$ which is a cusp form. Then*

(4.2) $$\sum\nolimits_{u' \in G^1_r(k)} \chi(u')/|G^1_r(k)|=(-1)^{|\Delta|} \qquad \textit{or } 0,$$

where $G^1_r(k)=\{$regular unipotent elements in $G(k)\}$.

Remark. It is conjectured by I. G. Macdonald (see [1; Part C]) and E. Bannai and H. Enomoto (see p. 148 of J. Alg. 20 (1972)) that analogues of (4.1) and (4.2) are also valid for a general irreducible character χ.

Proof of Proposition 1. Let Θ be the function on $G(k)$ defined by (1.1). By the Frobenius reciprocity theorem we have

$$(\chi, \Theta)_{G(k)} = \sum_{\delta \subset \Delta} (-1)^{|\delta|} (\chi \,|\, P_\delta(k), \theta_\delta)_{P_\delta(k)},$$

where $\chi \,|\, P_\delta(k)$ is the restriction of χ to $P_\delta(k)$. Using the Levi decomposition $P_\delta(k) = G_\delta(k) U_\delta(k)$ we have

$$(\chi \,|\, P_\delta(k), \theta_\delta)_{P_\delta(k)} = \sum_{y \in G_\delta^1(k)} \{\sum_{u \in U_\delta(k)} \chi(mu)\} / |P_\delta(k)|.$$

By the definition of cusp forms this equals to 0 unless $\delta = \Delta$. For $\delta = \Delta$ this is

$$\sum_u \chi(u) / |G(k)|.$$

On the other hand, Lemma 1 (b) leads to

$$(\chi, \Theta)_{G(k)} = \chi(1) q^m / |G(k)|.$$

Hence

$$(-1)^{|\Delta|} \sum_u \chi(u) = \chi(1) q^m.$$

This proves the proposition.

To prove Proposition 2, we need the following

Lemma 3. *Let χ be an irreducible character of $G(k)$ which is a cusp form. Then*

$$(\chi, i[\psi_\delta])_{G(k)} = 0 \qquad \textit{for all } \delta \subset \Delta$$

or $(\chi, i[\psi_\Delta])_{G(k)} = 1$ and $(\chi, i[\psi_\delta])_{G(k)} = 0$ for all $\delta \subsetneqq \Delta$.

Proof. This is a consequence of the following two facts:

(a) (S. I. Gelfand: Math. USSR. Sb., **12** (1970), P. 21) Let χ be a cusp form on $G(k)$ and $(\chi, i[\psi_\delta])_{G(k)} \neq 0$ for some $\delta \subset \Delta$. Then $(\chi, \psi_\Delta)_{G(k)} \neq 0$ and $(\chi, \psi_\delta)_{G(k)} = 0$ for all $\delta \subsetneqq \Delta$. (Actually, Gelfand proved this only for $G = GL_n$. But his proof works also for the general case.)

(b) (I. M. Gelfand and M. I. Graev, R. Steinberg [8; § 14]) If $(\chi, i[\psi_\Delta])_{G(k)} \neq 0$, $(\chi, i[\psi_\Delta])_{G(k)} = 1$.

Proof of Proposition 2. Let Ψ be the function on $G(k)$ defined by (1.2). By Lemma 2 we have

$$(\chi, \Psi)_{G(k)} = \sum_{u'} \chi(u') q^{|\Delta|} / |G(k)|.$$

On the other hand, by Lemma 3 we have

$$(\chi, \Psi)_{G(k)} = (-1)^{|\Delta|} \qquad \text{or } 0.$$

Hence

$$\sum_{u'} \chi(u') q^{|\Delta|} / |G(k)| = (-1)^{|\Delta|} \qquad \text{or } 0.$$

This combined with [1; Part E, III, 1.20] gives the required result.

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