149. On Quasi-primality of Submodules and of Ideals in Rings

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W. E. Barnes [1] has given for rings a theory of the representation of an ideal as an intersection of primal ideals, and showed that, in any short reduced representation of an ideal by primal ideals with prime adjoints, the adjoints and the number of primal components are uniquely determined. As is well-known, in that case there exist no containment relations among the prime adjoints. In order to generalize the above results, we shall consider a representation of a submodule by quasi-primal submodules, and as a special case we obtain that any two short reduced representations of an ideal by primal ideals have the same number of primal components and the same McCoy's radicals of their adjoints in pairs, if there exist no containment relations among the McCoy's radicals of the adjoints of primal components.

Throughout this note, R is a noncommutative ring whose unity does not necessarily exist, and M is a right R-module. The term *ideals* mean *two-sided ideals*, and (x) means the principal ideal by an element x of R. For a subset S of R, we set $\bar{S}=\{x\in R\mid (x)^n\subseteq S$ for some positive integer $n\}$, and set $\tilde{S}=\bigcap_i\{P_i\mid P_i$ is a prime ideal and $P_i\supseteq S\}$. Hence $\tilde{S}$ is an ideal. For convenience, even if a subset S of R is not an ideal, $\tilde{S}$ is called the *McCoy's radical* of S. For all ordinal numbers α we define $\bar{S}^{(\alpha)}$ by induction as follows: $\bar{S}^{(1)}=\bar{S}$, if α is not a limit ordinal then $\bar{S}^{(\alpha)}=\overline{\bar{S}^{(\alpha-1)}}$, and if α is a limit ordinal then $\bar{S}^{(\alpha)}=\bigcup_{\beta<\alpha}\bar{S}^{(\beta)}$.

Definition 1. Let S be a subset of R. If $\bar{S}^{(\alpha)}$ is an ideal for some ordinal number α, S is called a *quasi-ideal.*

Definition 2. A submodule N of M is called a *primal submodule* if its *adjoint subset* $N^a=\{x\in R\mid N\colon x\supsetneqq N\}$ is an ideal, where $N\colon x$ means the submodule $\{m\in M\mid mRx\subseteq N\}$. A submodule N of M is called a *quasi-primal submodule* if the adjoint subset N^a is a quasi-ideal. Evidently a primal submodule is quasi-primal.

Lemma 1. *If an ideal A of R is contained in the set-union of finitely many semi-prime ideals Q_i, then A is contained in one of the Q_i.*

Proof. Suppose that $A\not\subseteq Q_i$ for every i, then there exist prime ideals P_i such that $P_i\supseteq Q$ and $P_i\not\supseteq A$. Hence $A\subseteq\bigcup_{i=1}^n Q_i\subseteq\bigcup_{i=1}^n P_i$. This contradicts the well-known McCoy's result.

From [2], we obtained the following:

Lemma 2. *For an ideal A, $\overline{A}=\overline{A}^{(2)}$ if and only if $\overline{A}=\tilde{A}$.*

Lemma 3. *For subsets B, C of R, $B\subseteq\tilde{C}$ implies that $\overline{B}^{(\alpha)}\subseteq\tilde{C}$ for any ordinal number α.*

Proof. Suppose that $(x)^n\subseteq B$ for some positive integer n. Since $B\subseteq\tilde{C}=\bigcap_i\{P_i\,|\,P_i \text{ is prime}\}$, $(x)^n\subseteq P_i$ for every i, hence $(x)\subseteq P_i$. Thus we have $\overline{B}\subseteq\tilde{C}$. We shall prove the lemma by induction as follows: If α is not a limit ordinal, we have $\overline{B}^{(\alpha)}=\overline{\overline{B}^{(\alpha-1)}}\subseteq\overline{\tilde{C}}=\tilde{C}$, and if α is a limit ordinal, we have $\overline{B}^{(\alpha)}=\bigcup_{\beta<\alpha}\overline{B}^{(\beta)}\subseteq\tilde{C}$.

Lemma 4. *For a quasi-ideal A of R, there exists an ordinal number α sucn that $\overline{A}^{(\alpha)}=\tilde{A}$.*

Proof. By Lemma 3 $\overline{A}^{(\gamma)}\subseteq\tilde{A}$ for each ordinal number γ. Since the $\overline{A}^{(\gamma)}$ are well ordered and the set-union of every subset of them is again an $\overline{A}^{(\gamma)}$, by Zorn's Lemma they are all contained in a maximal one, $\overline{A}^{(\alpha)}$. On the other hand, since A is a quasi-ideal we may suppose that $\overline{A}^{(\alpha)}$ is an ideal and that $\overline{A}^{(\alpha+1)}=\overline{A}^{(\alpha)}$. Hence by Lemma 2, $\overline{\overline{A}^{(\alpha)}}=\widetilde{\overline{A}^{(\alpha)}}\supseteq\tilde{A}$. Thus we obtain $\overline{A}^{(\alpha)}=\tilde{A}$.

Corollary. *For any ideal A of R, $\overline{A}^{(\alpha)}=\tilde{A}$ for some ordinal number α.*

Now for any subset S of R, we set $\hat{S}$ to be the set-union of $\overline{S}^{(\gamma)}$ for all ordinal numbers γ. Similarly to the proof of Lemma 4, we have $\hat{S}=\overline{S}^{(\alpha)}$ for some ordinal number α.

Lemma 5. *For finitely many quasi-ideals Q_i, let $S=Q_1\cup\cdots\cup Q_n$, then $\hat{S}=\overline{S}^{(\alpha)}=\tilde{Q}_1\cup\cdots\cup\tilde{Q}_n$ for some ordinal number α.*

Proof. Suppose that $(x)^n\subseteq S\subseteq\tilde{Q}_1\cup\cdots\cup\tilde{Q}_n$, then by Lemma 1 $(x)^n\subseteq\tilde{Q}_i$ for some i, hence $x\in(x)\subseteq\tilde{Q}_i$. Thus we have $\overline{S}\subseteq\tilde{Q}_1\cup\cdots\cup\tilde{Q}_n$. We shall prove the lemma by induction as follows: If γ is not a limit ordinal, $\overline{S}^{(\gamma)}=\overline{\overline{S}^{(\gamma-1)}}\subseteq\overline{\tilde{Q}_1\cup\cdots\cup\tilde{Q}_n}$. Now repeating the above demonstration, we obtain $\overline{Q_1\cup\cdots\cup\tilde{Q}_n}\subseteq\tilde{\tilde{Q}}_1\cup\cdots\cup\tilde{\tilde{Q}}_n=\tilde{Q}_1\cup\cdots\cup\tilde{Q}_n$. If γ is a limit ordinal, $\overline{S}^{(\gamma)}=\bigcup_{\beta<\gamma}\overline{S}^{(\beta)}\subseteq\tilde{Q}_1\cup\cdots\cup\tilde{Q}_n$. Conversely, for any ordinal number γ and for any Q_i, $\overline{S}^{(\gamma)}\supseteq\overline{Q}_i^{(\gamma)}$. By Lemma 4, $\overline{S}^{(\delta)}\supseteq\tilde{Q}_i$ for some ordinal number δ. Thus we obtain $\hat{S}\supseteq\tilde{Q}_1\cup\cdots\cup\tilde{Q}_n$. Hence $\overline{S}^{(\alpha)}=\tilde{Q}_1\cup\cdots\cup\tilde{Q}_n$ for some ordinal number α.

Definition 3. A representation

$$N=N_1\cap\cdots\cap N_n \tag{1}$$

of submodule N of M as the intersection of submodules N_i of M is called *reduced* if no N_i can replaced by a proper large submodule. If (1) is a reduced representation of N by quasi-primal submodules N_i, and is such that $N_i\cap N_j$ is not quasi-primal if $i\neq j$, it is called a *short reduced representation of N by quasi-primal submodules.*

Theorem 1. *Let $N=Q_1\cap\cdots\cap Q_n$ be a reduced representation of a submodule N of M by quasi-primal submodules. For an ideal A of*

R, $A\subseteq\hat{N}^a$ if and only if $A\subseteq\tilde{Q}_i^a$ for some i.

Proof. By the definition of the reduced representation, we obtain $N^a=Q_1^a\cup\cdots\cup Q_n^a$ and every Q_i^a is a quasi-ideal. Hence by Lemma 5, $\hat{N}^a=\tilde{Q}_1^a\cup\cdots\cup\tilde{Q}_n^a$. Thus for any ideal A, $A\subseteq\hat{N}^a$ if and only if $A\subseteq\tilde{Q}_i^a$ for some i by Lemma 1.

Theorem 2. *Let*

$$N=Q_1\cap\cdots\cap Q_n \tag{1}$$

be a reduced representation of a submodule N of M by quasi-primal submodules, then N has a short reduced representation by quasi-primal submodules such that there exist no containment relations among the $\tilde{Q}_i^a$.

Proof. If there exist Q_i and Q_j such that $\tilde{Q}_i^a\supseteq\tilde{Q}_j^a$, set $Q=Q_i\cap Q_j$. Since (1) is reduced, $Q=Q_i\cap Q_j$ is reduced. By Lemma 5 $\overline{Q}^{(\alpha)}=\tilde{Q}_i^a\cup\tilde{Q}_j^a=\tilde{Q}_i^a$ for some ordinal number α. Since $\tilde{Q}_i^a$ is an ideal, Q is quasi-primal. Conversely if there exist no containment relations among the $\tilde{Q}_i^a$, set $Q=Q_{i_1}\cap\cdots\cap Q_{i_t}$ for any $Q_{i_1},\cdots,Q_{i_t}$. Again we obtain $\hat{Q}^a=\tilde{Q}_{i_1}^a\cup\cdots\cup\tilde{Q}_{i_t}^a$ by Lemma 5. If Q is quasi-primal, by Lemma 4 $\hat{Q}^a=\tilde{Q}^a$ is an ideal. Hence by Theorem 1 we may assume that $\hat{Q}^a\subseteq\tilde{Q}_{i_1}$ and for any i_k $\tilde{Q}_{i_k}^a\subseteq\hat{Q}^a\subseteq\tilde{Q}_{i_1}^a$, which is a contradiction.

Corollary. *Let*

$$N=Q_1\cap\cdots\cap Q_n \tag{2}$$

be a reduced representation by quasi-primal submodules. Then (2) is a short representation if and only if there exist no containment relations among $\tilde{Q}_i^a$.

Theorem 3. *In any short reduced representation of submodule N by quasi-primal submodules, the McCoy's radicals of adjoints and the number of quasi-primal components are uniquely determined.*

Proof. Let $N=Q_1\cap\cdots\cap Q_n=P_1\cap\cdots\cap P_m$ be any two representation of N by quasi-primal submodules. For each i $\tilde{Q}_i^a\subseteq\hat{N}^a$, and by Theorem 1 there exists j such that $\tilde{Q}_i^a\subseteq\tilde{P}_j^a$. Similarly we obtain $\tilde{P}_j^a\subseteq\tilde{Q}_k^a$ for some k. Hence we have $n=m$ and for any i $\tilde{Q}_i^a=\tilde{P}_j^a$ for some order.

Corollary. *In any short reduced representation of N by primal submodules such that there exist no containment relations among the McCoy's radicals of their adjoints, then the McCoy's radical of adjoints and the number of primal components are uniquely determined.*

Proof. By Corollary to Theorem 2 and Theorem 3.

Now, especially let N be an ideal of R, the corollary is a generalization of Theorem 5, mentioned in the introduction, of W. E. Barnes [1].

At last, we shall show that a quasi-primal submodule is primal if the following condition holds:

Axiom D of L. Lesieur and R. Croisot [3]—The sets of right residues and left residues of all submodules of M satisfies the ascending chain condition.

Theorem 4. *If axiom D holds, then a quasi-primal submodule N is primal.*

Proof. Let N be a quasi-primal submodule. By Theorem 4.1 of [3], we obtain a short reduced representation, in the sense of [3], $N=Q_1\cap\cdots\cap Q_n$ by primal submodules with prime adjoints such that there exist no containment relations among Q_i^a. Hence we have $N^a=Q_1^a\cup\cdots\cup Q_n^a$, and by Lemma 5 $\hat{N}^a=Q_1^a\cup\cdots\cup Q_n^a$. Since N is quasi-primal $\hat{N}^a$ is an ideal. Thus by Lemma 1 $n=1$, hence N is primal.

References

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