39. On G_δ-Sets in the Product of a Metric Space and a Compact Space. I

By Jun-iti NAGATA*)
Department of Mathematics, University of Pittsburgh

(Comm. by Kinjirô KUNUGI, M. J. A., March 12, 1973)

We have proved in [8] that a topological space is paracompact (Hausdorff) and M if and only if it is homeomorphic to a closed set of the product of a metric space and a compact Hausdorff space. A similar characterization for general M-spaces may be obtained, but it is still an open question whether 'M-space' is characterized as a closed set in the product of a metric space and a countably compact space (see [9]). In this brief note we are going to turn our attention to G_δ-sets in the product of a metric space and a compact space. Although we are not successful yet in finding an internal characterization of those sets, they seem deeply related with A. V. Arhangelskii's p-spaces (see [1]) as will be seen in the following discussion. All spaces in this paper are at least Hausdorff, and all maps (=mappings) are continuous. As for the concept of M-space (due to K. Morita) the reader is referred to [4]. For general terminologies and symbols in general topology (see [6]).

Theorem 1. *An M-space X is homemorphic to a G_δ-set in the product of a metric space and a compact Hausdorff space if and only if it is a p-space.*

Proof. It is known that the product of a metric space and a compact Hausdorff space is paracompact and p, and it is also easy to see that every G_δ-set of a p-space is p. Therefore we shall prove only the 'if' part of the theorem. Assume that X is M and p at the same time. Then by Morita's theorem [4] there is a quasi-perfect map f from X onto a metric space Y. (Namely f is closed and continuous, and $f^{-1}(y)$ is countably compact for each $y \in Y$.) By D. Burke's theorem [3] there is a sequence $\mathcal{V}_1, \mathcal{V}_2, \cdots$ of open covers of X such that

(i) if $x \in V_i \in \mathcal{V}_i$, $i=1,2,\cdots$, then $K=\bigcap_{i=1}^{\infty} \bar{V}_i$ is compact,

(ii) for every open set U containing K, there is k for which $\bigcap_{i=1}^{k} \bar{V}_i \subset U$.

We may assume without loss of generality that each $\mathcal{V}_i$ consists of cozero open sets (=complements of zero sets of real-valued continuous functions defined on X), because X is a Tychonoff space (which is implied by the fact that X is p).

---

*) Supported by NSF Grant GP29401.

Now we define a map g from X into the product space $\beta X \times Y$ of the Stone-Čech compactification βX of X and Y as follows:

$$g(x)=(x, f(x)), \qquad x \in X.$$

It is quite easy to see that g is a topological map from X onto $g(X)$. So all we have to prove is that $g(X)$ is a G_δ-set in $\beta X \times Y$. Now note that we may regard βX as the set of all maximal filters consisting of zero sets in X (see [6]), and put

$$C=\{(z, y) \in \beta X \times Y \mid f^{-1}(y) \in z\} .$$

Then we can prove that C is a G_δ-set in $\beta X \times Y$. (Actually it is a closed G_δ-set.) For each cozero open set V of X we define an open set $V^\sim$ of βX by $V^\sim=\{z \in \beta X \mid X-V \notin z\}$. We also denote by $S_n(y)$ the $1/n$-nbd (=neighborhood) of a point y of Y, where n is a natural number. For each point y of Y and for each natural number n, let

$$M_n(y)=(f^{-1}(S_n(y)))^\sim \times S_n(y).$$

Then $M_n(y)$ is an open nbd of each $(z, y) \in C$. Furthermore we put

$$M_n=\cup\{M_n(y) \mid y \in Y\}$$

to obtain an open set M_n of $\beta X \times Y$ satisfying $M_n \supset C$. Now we claim that $C=\bigcap_{n=1}^\infty M_n$. To prove it, let $(z', y') \in \beta X \times Y - C$. Then $f^{-1}(y') \notin z'$, i.e. there is a set $F \in z'$ such that $F \cap f^{-1}(y')=\emptyset$. Hence $y' \notin f(F)$ in Y. Since $f(F)$ is a closed set, $S_n(y') \cap f(F)=\emptyset$ for some natural number n. This implies that $(z', y') \notin M_{3n}(y)$ for each $y \in Y$, and accordingly $(z', y') \notin M_{3n}$. Because if $\rho(y', y) \geqq 1/3n$, then $y' \notin S_{3n}(y)$ implying that $(z', y') \notin M_{3n}(y)$. If $\rho(y', y)<1/3n$, then $S_{3n}(y) \cap f(F)=\emptyset$ in Y, which implies that $f^{-1}(S_{3n}(y)) \cap F=\emptyset$ in X, and hence $X-f^{-1}(S_{3n}(y)) \in z'$. Therefore $z' \notin (f^{-1}(S_{3n}(y)))^\sim$, and $(z', y') \notin M_{3n}(y)$ follows. Thus in any case $(z', y') \notin M_{3n}(y)$ is proved. Finally let us prove that $g(X)$ is a G_δ-set in C. It is obvious that $g(X) \subset C$. Now we define subsets P_n of C by

$$P_n=\{(z, y) \in C \mid z \in V^\sim \text{ for some } V \in \mathcal{V}_n\}, \qquad n=1, 2, \cdots.$$

It is again obvious that $g(X) \subset P_n$, $n=1, 2, \cdots$. Since

$$P_n=C \cap [\cup\{V^\sim \times Y \mid V \in \mathcal{V}_n\}],$$

it is an open set of C. Thus all we have to show is that $\bigcap_{n=1}^\infty P_n \subset g(X)$. Let $(z, y) \in \bigcap_{n=1}^\infty P_n$; then there are $V_n \in \mathcal{V}_n$, $n=1, 2, \cdots$ for which $z \in \bigcap_{n=1}^\infty V_n^\sim$. Hence there are $F_n \in z$ such that $F_n \subset V_n$, $n=1, 2, \cdots$. Since $(z, y) \in C$, $f^{-1}(y) \in z$. Therefore the collection $\{f^{-1}(y), F_n \mid n=1, 2, \cdots\}$ has f.i.p. (=the finite intersection property). Since $f^{-1}(y)$ is countably compact, there is $x \in f^{-1}(y) \cap (\bigcap_{n=1}^\infty F_n)$. Thus $x \in V_n \in \mathcal{V}_n$, $n=1,2, \cdots$. Hence from the property of $\mathcal{V}_n$ it follows that $K=\bigcap_{n=1}^\infty \overline{V}_n$ is compact, and $\{K\} \cup z$ has f.i.p. Because if not, then $K \cap F=\emptyset$ for some $F \in z$, and hence $(\bigcap_{n=1}^k \overline{V}_n) \cap F=\emptyset$ for some k. Since $F_n \subset V_n$, this implies that $(\bigcap_{n=1}^k F_n) \cap F=\emptyset$, which is a contradiction. Since K is compact, and z is a maximal filter of zero sets, we can conclude that z converges to a point p of K. (In order words z is the filter consisting of all zero sets containing p.) Thus $z=p$ in βX and $p \in f^{-1}(y)$, where the latter follows from the fact $f^{-1}(y) \in z$. Hence $(z, y)=g(p) \in g(X)$, proving that $\bigcap_{n=1}^{\infty} P_n \subset g(X)$. After all we have proved that $g(X)$ is G_δ in C, which is G_δ in $\beta X \times Y$. Thus $g(X)$ is G_δ in $\beta X \times Y$.

Corollary. *Every paracompact M-space is homeomorphic to a closed G_δ-set in the product of a metric space and a compact Hausdorff space.*

Proof. In [8] we have proved that $g(X)$ in the proof of Theorem 1 is a closed set of $\beta X \times Y$ if X is paracompact M. Hence this corollary follows.

Definition. A topological space is called a *G_δ-space* if it is homeomorphic to a G_δ-set in the product of a metric space and a compact Hausdorff space. Every metric space as well as every topologically complete space in the sense of E. Čech is a G_δ-space, and every G_δ-space is a p-space.

Problem 1. Is every p-space a G_δ-space?

Although a positive answer to this problem means a beautiful characterization of p-space as well as of G_δ-space, the answer will be more likely 'no'. Then the next question is

Problem 2. Give an internal characterization of G_δ-space.

Theorem 1 looks like a suggestion that the answer for Problem 2 may be 'M and p'. But D. Burke [3] gave an example of a locally compact Hausdorff space which is not $w\Delta$ (accordingly not M). Therefore a G_δ-space is not necessarily M. However, K. Morita [3] proved that if a G_δ-set of a normal M-space is the intersection of countably many open F_σ-sets, then it is M. Thus we have

Problem 3. Is every M and p-space homeomorphic to a G_δ-set S in the product of a metric space and a compact Hausdorff space such that S is the intersection of countably many open F_σ-sets? (It is unknown if $g(X)$ in the proof of Theorem 1 is such a G_δ-set.)

Theorem 2. *A metacompact Tychonoff space Y is a p-space if and only if there is a G_δ-space X and a compact open map f from X onto Y.* (A topological space is called metacompact if for every open cover there is a point-finite open refinement.)

Proof. The 'if' part follows from K. Nagami's theorem [5] which implies that the compact open image of a p-space is p provided it is metacompact and Tychonoff. The 'only if' part will be proved as follows. Let Y be metacompact and p. Then Y has a sequence $\mathcal{V}_1, \mathcal{V}_2, \cdots$ of open covers satisfying the condition of Burke's theorem. We may assume that each $\mathcal{V}_i$ is a point-finite open cover consisting of cozero open sets. Let $\mathcal{V}_i=\{V_\alpha^i \mid \alpha \in A\}$, $i=1, 2, \cdots$, and define a subset X of the product space $Y \times N(A)$ as follows, where $N(A)$ denotes the Baire's zerodimensional space, i.e. the countable product of the copies of the discrete space A.

$$X=\{(y,(\alpha_1,\alpha_2,\cdots))\in Y\times N(A)\mid y\in V^1_{\alpha_1}\cap V^2_{\alpha_2}\cap\cdots\}.$$

We define a map f from X onto Y by

$$f(y,(\alpha_1,\alpha_2,\cdots))=y \qquad \text{for } (y,(\alpha_1,\alpha_2,\cdots))\in X.$$

It is obvious that f is a continuous open map. The compactness of f is obtained by a rather routine method of proof (see, for example, [7]), because each $\mathcal{V}_i$ is point-finite. Now put

$$M=\{(\alpha_1,\alpha_2,\cdots)\in N(A)\mid V^1_{\alpha_1}\cap V^2_{\alpha_2}\cap\cdots\neq\emptyset\}.$$

Then we can prove that X is a G_δ-set in $\beta Y\times M$. For each $(\alpha_1,\alpha_2,\cdots)\in M$, we put

$$P(\alpha_1,\alpha_2,\cdots,\alpha_i)=(V^1_{\alpha_1}\cap\cdots\cap V^i_{\alpha_i})^{\sim}\times N(\alpha_1,\cdots,\alpha_i),$$

where $N(\alpha_1,\cdots,\alpha_i)=\{(\beta_1,\beta_2,\cdots)\in M\mid \beta_1=\alpha_1,\cdots,\beta_i=\alpha_i\}$. Furthermore we let

$$P_i=\cup\{P(\alpha_1,\cdots,\alpha_i)\mid(\alpha_1,\alpha_2,\cdots)\in M\}.$$

Then each P_i is an open set of $\beta Y\times M$ such that $P_i\supset X$. To prove $X=\bigcap_{i=1}^\infty P_i$, let $p=(z,(\alpha_1,\alpha_2,\cdots))\in\beta Y\times M-X$. If $z\in Y$, then $z\notin V^1_{\alpha_1}\cap\cdots\cap V^n_{\alpha_n}$ for some n. Therefore $p\notin P(\alpha_1,\cdots,\alpha_n)$. Since it is obvious that $p\notin P(\beta_1,\cdots,\beta_n)$ for $(\beta_1,\cdots,\beta_n)$ different from $(\alpha_1,\cdots,\alpha_n)$, $p\notin P_n$ follows. If $z\in\beta Y-Y$, then there is $F\in z$ such that $F\cap(\bigcap_{i=1}^\infty \overline{V}^i_{\alpha_i})=\emptyset$, because the compactness of $\bigcap_{i=1}^\infty\overline{V}^i_{\alpha_i}$ follows from that $\bigcap_{i=1}^\infty V^i_{\alpha_i}\neq\emptyset$. Hence $F\cap(\bigcap_{i=1}^n\overline{V}^i_{\alpha_i})=\emptyset$ for some n. Thus $Y-\bigcap_{i=1}^n V^i_{\alpha_i}\in z$, implying that $z\notin(V^1_{\alpha_1}\cap\cdots\cap V^n_{\alpha_n})^{\sim}$. Namely $p\notin P(\alpha_1,\cdots,\alpha_n)$. Again $p\notin P(\beta_1,\cdots,\beta_n)$ is obvious if $(\beta_1,\cdots,\beta_n)\neq(\alpha_1,\cdots,\alpha_n)$, and hence $p\notin P_n$. After all $X=\bigcap_{i=1}^\infty P_i$ is concluded.

Problem 4. Characterize the compact open images of G_δ-spaces.

References

[1] A. V. Arhangelskii: Mappings and spaces. Russian Mathematical Surveys, **21**, 115–162 (1966).

[2] D. K. Burke: On p-spaces and $w\Delta$-spaces. Pacific J. Math., **35**, 285–296 (1970).

[3] K. Morita: Some properties of M-spaces. Proc. Japan Acad., **43**, 869–872 (1967).

[4] ——: A survey of the theory of M-spaces. General Topology and its Applications, **1**, 49–55 (1971).

[5] K. Nagami: Minimal class generated by open compact and perfect mappings (to appear).

[6] J. Nagata: Modern General Topology. Amsterdam-Groningen (1968).

[7] ——: Mappings and M-spaces. Proc. Japan Acad., **45**, 140–144 (1969).

[8] ——: A note on M-space and topologically complete space. Proc. Japan Acad., **45**, 541–543 (1969).

[9] ——: Problems on generalized metric spaces. Proc. Emory Topology Conference, 63–70 (1970).