52. On Dual Multiplicative Functionals

By Sadao SUGITANI
Department of Mathematics, Osaka University

(Comm. by Kôsaku YOSIDA, M. J. A., April 12, 1973)

§1. Introduction and the results. Let X and $\hat{X}$ be standard (Markov) processes which are in duality relative to some measure. Let (M_t) be a multiplicative functional of X. R. K. Getoor [2] has proved the existence and uniqueness of a dual (exact) multiplicative functional $(\hat{M}_t)$ of (M_t) under the hypothesis of absolute continuity. The purpose of this note is to extend the results of Getoor to the general case without the above hypothesis.

We begin with some notation and terminology of Markov processes, following the book of Blumenthal-Getoor [1]. Let

$$X=(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x) \quad \text{and} \quad \hat{X}=(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{X}_t, \hat{\theta}_t, \hat{P}^x)$$

be standard processes with the same state space E. Their semi-groups and resolvents are, respectively, denoted by $(P_t), (\hat{P}_t), (U_\alpha)$ and $(\hat{U}_\alpha)$. We shall say that X and $\hat{X}$ are *in duality* relative to a Radon measure ξ, if

$$\int f(x)U_\alpha g(x)\xi(dx)=\int \hat{U}_\alpha f(x)g(x)\xi(dx) \qquad \alpha>0$$

or equivalently

$$\int f(x)P_t g(x)\xi(dx)=\int \hat{P}_t f(x)g(x)\xi(dx) \qquad t\geqslant 0$$

for any nonnegative and universally measurable functions f and g. We *do not assume* the resolvents (U_α) and $(\hat{U}_\alpha)$ are absolutely continuous relative to ξ. In the following the integral $\int f(x)g(x)\xi(dx)$ is written as (f, g). As usual $\mathcal{E}$ (resp. $\mathcal{E}^*$) is a σ-algebra of Borel (resp. universally measurable) subsets of E, and $f \in \mathcal{E}$ (resp. $\mathcal{E}^*$) means that f is $\mathcal{E}$ (resp. $\mathcal{E}^*$)-measurable.

Let (M_t) be a multiplicative functional (abbreviated as *MF*) of X. In this paper all *MF*'s are assumed to be right continuous, decreasing, and to satisfy $0\leqslant M_t\leqslant 1$. The semigroup and resolvent generated by (M_t) are denoted by (Q_t) and (V_α):

$$Q_t f(x)=E^x[f(X_t)M_t], \qquad t\geqslant 0,$$

$$V_\alpha f(x)=E^x\int_0^\infty e^{-\alpha t}f(X_t)M_t dt, \qquad \alpha>0.$$

A $MF(\hat{M}_t)$ of $\hat{X}$ is said to be a *dual multiplicative functional* of (M_t) if the relation

$$(f, V_\alpha g)=(\hat{V}_\alpha f, g) \qquad \alpha>0$$

or equivalently

$$(f, Q_t g) = (\hat{Q}_t f, g) \qquad t \geqslant 0$$

is satisfied for any $f, g \in \mathcal{E}^*_+$.

We then have

Theorem 1. *For any MF (M_t) of X, there exists a dual exact MF $(\hat{M}_t)$.*

For the uniqueness we need the following definition due to Fukushima. A subset A of E is said to be an *almost polar set* if there exist a Borel set B such that $B \supset A$ and $P^x(T_B < \infty) = 0$ ξ-a.e. x. where $T_B(\omega) = \inf\{t > 0; X_t(\omega) \in B\}$. An almost polar set with respect to the dual process $\hat{X}$ is called an *almost copolar set.*

Theorem 2. *Let $(\hat{M}_t)$ and $(\hat{N}_t)$ be dual exact MF's of (M_t), then,*

$$\hat{P}^x(\hat{M}_t = \hat{N}_t, t < \zeta) = 1$$

is valid except on an almost copolar set.

§2. Proofs. Proof of Theorem 1. 1°. Since $V_\alpha \leqslant U_\alpha$ there exists a function $v_\alpha(x, y) \in \mathcal{E}^* \times \mathcal{E}^*$ such that $0 \leqslant v_\alpha \leqslant 1$ and $V_\alpha f(x) = \int v_\alpha(x, y) f(y) U_\alpha(x, dy)$. Define

$$\hat{V}_\alpha f(x) =: \int v_\alpha(y, x) f(y) \hat{U}_\alpha(x, dy).$$

It is obvious that

$$(f, V_\alpha g) = (\hat{V}_\alpha f, g). \tag{1}$$

This family $(\hat{V}_\alpha)$ may not be a true resolvent. We have to show that $(\hat{V}_\alpha)$ can be modified into a true resolvent which is exactly subordinate to $(\hat{U}_\alpha)$.

2°. *There exists $Y_0 \in \mathcal{E}^*$ such that $\xi(Y_0) = 0$ and $\hat{V}_\alpha(x, \cdot) - \hat{V}_\beta(x, \cdot) + (\alpha - \beta)\hat{V}_\alpha \hat{V}_\beta(x, \cdot) = 0$ for any rationals $\alpha, \beta > 0$ and every $x \in Y_0$.*

Let $\{f_n\}_{n \geqslant 1}$ be a countable dense subset of C_K (=the set of all continuous functions with compact support), and $Y_0 = \bigcup_{\substack{\alpha, \beta > 0 \\ \text{rational}}} \bigcup_{n \geqslant 1} Y_{n,\alpha,\beta}$, where $Y_{n,\alpha,\beta} = \{x : \hat{V}_\alpha f_n(x) - \hat{V}_\beta f_n(x) + (\alpha - \beta)\hat{V}_\alpha \hat{V}_\beta f_n(x) \neq 0\}$. It is sufficient to show that for any n, α and β, $\xi(Y_{n,\alpha,\beta}) = 0$. Let f be the indicator of the set $\{\hat{V}_\alpha f_n - \hat{V}_\beta f_n + (\alpha - \beta)\hat{V}_\alpha \hat{V}_\beta f_n > 0\}$. Using (1), $((\hat{V}_\alpha - \hat{V}_\beta + (\alpha - \beta)\hat{V}_\alpha \hat{V}_\beta) f_n, f) = (f_n, (V_\alpha - V_\beta + (\alpha - \beta) V_\beta V_\alpha) f) = 0$, which proves that $\xi(Y_{n,\alpha,\beta}) = 0$.

3°. *There exists $(Y_n)_{n \geqslant 0}$ in $\mathcal{E}^*$ such that $\xi(Y_n) = 0$ and $\hat{V}_\alpha(x, \bigcup_{k=0}^{n-1} Y_k) = 0$ for any $x \in Y_n^c$ and rational $\alpha > 0$.*

Construct Y_n by $\bigcup_{\substack{\alpha > 0 \\ \text{rational}}} \{x : \hat{V}_\alpha(x, \bigcup_{k=0}^{n-1} Y_k) > 0\}$. By (1), for each α,

$$\alpha \xi \hat{V}_\alpha \left(\bigcup_{k=0}^{n-1} Y_k\right) = \left(\alpha \hat{V}_\alpha \left(x, \bigcup_{k=0}^{n-1} Y_k\right), 1\right) = (I_{\cup_{k=0}^{n-1} Y_k}, \alpha V_\alpha 1)$$
$$\leq \xi\left(\bigcup_{k=0}^{n-1} Y_k\right) = 0.$$

which implies that the set $\{x : \hat{V}_\alpha(x, \bigcup_{k=0}^{n-1} Y_k) > 0\}$ has ξ-measure 0.

4°. Let $Z = \bigcup_{k=0}^{\infty} Y_k$. Obviously $\xi(Z) = 0$ and whenever $x \in Z^c$

(2) $$\hat{V}_{\alpha}f(x)-\hat{V}_{\beta}f(x)+(\alpha-\beta)\hat{V}_{\alpha}\hat{V}_{\beta}f(x)=0$$
for every rationals $\alpha,\beta>0$ and every bounded $f\in\mathcal{E}^*$. Take an arbitrary real $\alpha>0$. Then by $\hat{V}_{\beta}\leq\hat{U}_{\beta}$ and the equation (2), it follows that for every $x\in Z^c$, $f\in b\mathcal{E}^*$, $(\hat{V}_{\beta}f(x))$ has a limit as the rationals β tend to α. Therefore we can define for every $\alpha>0$

$$\hat{\hat{V}}_{\alpha}f(x)=\lim_{\substack{\beta\to\alpha\\ \beta\text{-rational}}}\hat{V}_{\beta}f(x) \qquad \text{if } x\in Z^c, f\in b\mathcal{E}^*$$
$$=0 \qquad \text{if } x\in Z, f\in b\mathcal{E}^*$$

It is casy to show that $(\hat{\hat{V}}_{\alpha})$ is a true resolvent satisfying $(\hat{\hat{V}}_{\alpha}f,g)=(f,V_{\alpha}g)$.

5°. $(\hat{U}_{\alpha}-\hat{\hat{V}}_{\alpha})f(x)$ is $\alpha-\hat{U}$ supermedian for $f\in b\mathcal{E}^*_+$. Let $\hat{W}_{\alpha}f(x)=:\lim_{\beta\to\infty}\beta\hat{U}_{\beta}\hat{\hat{V}}_{\alpha}f(x)$. Then $(\hat{W}_{\alpha})$ is a resolvent exactly subordinate to $(\hat{U}_{\alpha})$. (See Blumenthal-Getoor [1], p. 117.) Therefore, by a result of Meyer [3], there exists a unique semigroup $(\hat{Q}_t)$ such that, $0\leqslant\hat{Q}_tf\leqslant\hat{P}_tf$ for $f\in b\mathcal{E}^*_+$ and $t\geqslant 0$, $\lim_{t\to 0}\hat{Q}_t1=\hat{Q}_01$, and $\hat{W}_{\alpha}=\int_0^{\infty}e^{-\alpha t}\hat{Q}_tdt$. Therefore there exists an exact MF $(\hat{M}_t)$ of $\hat{X}$ generating $(\hat{Q}_t)$. This $(\hat{M}_t)$ is what we need, because for any $f,g\in C_K$,

$$(\hat{W}_{\alpha}f,g)=\lim_{\beta\to\infty}(\beta\hat{U}_{\beta}\hat{\hat{V}}_{\alpha}f,g)$$
$$=\lim_{\beta\to\infty}(\hat{\hat{V}}_{\alpha}f,\beta U_{\beta}g)=(\hat{\hat{V}}_{\alpha}f,g)=(f,V_{\alpha}g). \qquad \text{Q.E.D.}$$

Before proceeding to the proof of Theorem 2, we need a lemma.

Lemma. *Let $(\hat{M}_t)$ and $(\hat{N}_t)$ be dual exact MF's of (M_t) and let their resolvents be $(\hat{V}_{\alpha})$ and $(\hat{W}_{\alpha})$, respectively. Put*
$$B=\bigcup_{\alpha>0}\{x:\hat{V}_{\alpha}(x,\cdot)\neq\hat{W}_{\alpha}(x,\cdot)\}.$$
Then B is nearly Borel and almost copolar.

Proof. Let $\{f_n\}_{n\geqslant 1}$ be a countable dense subset in C_K. Then B can be written as $\bigcup_{\substack{\alpha>0\\ \text{rational}}}\bigcup_{n\geqslant 1}\{x:\hat{V}_{\alpha}f_n(x)\neq\hat{W}_{\alpha}f_n(x)\}$. From this representation, B is nearly Borel. If B is not almost copolar, then we may assume there exist α,n and k such that
$$B_{n,\alpha,k}=\left\{x:\hat{V}_{\alpha}f_k(x)>\hat{W}_{\alpha}f_k(x)+\frac{1}{n}\right\}$$
is not almost copolar. For simplicity we denote $B=B_{n,\alpha,k}$, $T=T_{B_{n,\alpha,k}}$, and $f=f_k$. Because $\hat{V}_{\alpha}f$ and $\hat{W}_{\alpha}f+1/n$ are cofinely continuous, we have
$$\hat{V}_{\alpha}f(\hat{X}_T)\geqslant\hat{W}_{\alpha}f(\hat{X}_T)+\frac{1}{n} \qquad \text{a.s. on } \{T<\infty\}.$$
Therefore $(\hat{P}^{\alpha}_B\hat{V}_{\alpha}f,1)\geqslant(\tilde{P}^{\alpha}_B(\hat{W}_{\alpha}f+1/n),1)$.

But we have
$$(\hat{P}^{\alpha}_B\hat{V}_{\alpha}f,1)=\lim_{\beta\to\infty}(\hat{P}^{\alpha}_B\beta\hat{U}_{\beta}\hat{V}_{\alpha}f,1)$$
$$=\lim_{\beta\to\infty}(\hat{P}^{\alpha}_B\hat{U}_{\alpha}(\beta+\beta(\alpha-\beta)\hat{U}_{\beta})\hat{V}_{\alpha}f,1)$$
$$=\lim_{\beta\to\infty}(f,V_{\alpha}(\beta+\beta(\alpha-\beta)U_{\beta})P^{\alpha}_BU_{\alpha}1).$$

Similarly

$$(\hat{P}_B^\alpha \hat{W}_\alpha f, 1) = \lim_{\beta\to\infty} (f, V_\alpha(\beta+\beta(\alpha-\beta)U_\beta) P_B^\alpha U_\alpha 1).$$

Therefore $(\hat{P}_B^\alpha 1, 1)=0$, so that $\xi\{x : \hat{P}_B^\alpha 1(x)>0\}=0$, showing that B is almost copolar. This contradicts the assumption that $B=B_{n,\alpha,k}$ is not almost copolar.

Proof of Theorem 2. Fukushima [4] has shown that for an almost polar set N, there exists a Borel set B such that $B\supset N, \xi(B)=0$ and $E-B$ is invariant with respect to the process X, i.e. $P^x(X_t \in E-B, 0\leqslant t<\zeta)=1$ for any $x\in E-B$. By this and the above lemma, there exists a Borel set B' such that $B'\supset B, \xi(B')=0$ and $E-B'$ is invariant with respect to the process $\hat{X}$. It is enough to show that

$$\{x : \hat{P}^x(\hat{M}_t=\hat{N}_t, t<\hat{\zeta})=1\}\supset E-B'.$$

If $x_0\in E-B'$, then $\hat{V}_\alpha(x_0, \cdot)=\hat{W}_\alpha(x_0, \cdot)$ by Lemma, and it follows that $\hat{Q}_t(x_0, \cdot)=\hat{R}_t(x_0, \cdot)$, where $\hat{Q}_t$ (resp. $\hat{R}_t$) is the semigroup generated by $(\hat{M}_t)$ (resp. $(\hat{N}_t)$). Since $E-B'$ is invariant with respect to $\hat{X}$, we obtain for any $F\in\hat{\mathcal{F}}_t, \hat{E}^{x_0}\{I_F(\omega)\hat{M}_t(\omega)\}=\hat{E}^{x_0}\{I_F(\omega)\hat{N}_t(\omega)\}$. This means $\hat{M}_t=\hat{N}_t$ a.s. $\hat{P}^{x_0}$ on $\{t<\hat{\zeta}\}$. By the right continuity of MF's, $\hat{P}^{x_0}(\hat{M}_t=\hat{N}_t, t<\hat{\zeta})=1$.

Remark. In the case of absolute continuity, we easily obtain the result of Getoor [2] as follows.

Proposition. *If both (U_α) and $(\hat{U}_\alpha)$ are absolutely continuous for ξ, then*

$$\hat{P}^x(\hat{M}_t=\hat{N}_t, t<\hat{\zeta})=1 \textit{ everywhere.}$$

($(\hat{M}_t)$ and $(\hat{N}_t)$ are those in Theorem 2.)

Proof. For any $f, g\in\mathcal{E}_+^*$ and $\alpha>0$ we obtain $(\hat{W}_\alpha f, g)=(f, V_\alpha g)=(\hat{V}_\alpha f, g)$. Then $\hat{W}_\alpha f(x)=\hat{V}_\alpha f(x)$ a.s. $\xi(x)$. By the absolute continuity $\beta\hat{U}_\beta\hat{W}_\alpha f(x)=\beta\hat{U}_\beta\hat{V}_\alpha f(x)$ for any $\beta>0$. Let $\beta\to\infty$, then $\hat{W}_\alpha f(x)=\hat{V}_\alpha f(x)$. Therefore $(\hat{M}_t)$ and $(\hat{N}_t)$ generate the same semigroups. This means that $(\hat{M}_t)=(\hat{N}_t)$ a.s.

References

[1] R. M. Blumenthal and R. K. Getoor: Markov Processes and Potential Theory. Academic Press, New York (1968).

[2] R. K. Getoor: Multiplicative functionals of dual processes. Ann. Inst. Fourier, **21** (2), 43–83 (1971).

[3] P. A. Meyer: Semigroupes en dualite. Séminaire de théorie du potentiel (Brelot-Choquet-Deny), 10 pages (1960/61). Paris.

[4] M. Fukushima: Almost polar sets and an ergodic theorem (to appear).