71. *Analyticity of Eigenvalues as Functions of Coupling Constant*

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Eigenvalues of the Hamiltonian in a quantum field theory can be considered as functions of the coupling constant. In the present paper, the analyticity of these functions is studied by using simple models.

1. Introduction. The total Hamiltonian H is written in the form $H=H^0+gH'$, where H^0 is the free Hamiltonian and gH' is the interaction term. In the eigenvalue equation

$$H\Psi=E\Psi \tag{1.1}$$

we call in question the analyticity of eigenvalues E which are functions of g. For this purpose, we use three simple models, the first two of them concern with the Lee model

$$V\rightleftarrows N\theta \tag{1.2}$$

and the third one is concerned with the reaction

$$\theta\rightleftarrows VN. \tag{1.3}$$

In the first case of reaction (1.2), all θ-particles have only one sort of momentum (*Case* 1). The second one of (1.2) is the usual Lee model (*Case* 2). In (1.3), θ-particles can have all sorts of momentum (*Case* 3). In these cases, the V- and N-particles have masses m_V and m_N, respectively, and are fixed in space. The rest mass of the Θ-particle is μ. The use of symbols V, N and a_k for the annihilation operators of V-, N- and θ-particles, respectively, is as usual [2]. In Case 1, symbol a is used in stead of a_k. The commutation or anticommutation relations among operators V, N, a_k etc. are also as usual [2]. In most cases, we assume the inequality $m_V>m_N+\mu$.

2. *Case* 1. We put $H^0=m_VV^*V+m_NN^*N+\omega a^*a$, and $H'=V^*Na+N^*Va^*$, where ω is a positive real constant. Let $u_V(0)$ and $u_V(1)$ be normalized eigenvectors of V^*V, and $u_N(0)$ and $u_N(1)$ be those of N^*N, and let $\mathfrak{H}_V$ and $\mathfrak{H}_N$ be two dimensional unitary spaces spanned by these sets of vectors, respectively. The infinite-dimensional Hilbert space spanned by normalized eigenvectors of a^*a is denoted by $\mathfrak{H}_\theta$. Then our basis space is the direct product of three spaces $\mathfrak{H}_V$, $\mathfrak{H}_N$ and $\mathfrak{H}_\theta$, i.e., $\mathfrak{H}(H^0)=\mathfrak{H}_V\otimes\mathfrak{H}_N\otimes\mathfrak{H}_\theta$. States in which the number of fermions is equal to one are written in the form

$$\Psi=\sum_{i+j=1}u_V(i)\otimes u_N(j)\otimes\phi_{ij}, \tag{2.1}$$

where ϕ_{ij}'s are vectors in $\mathfrak{H}_\theta$. We insert (2.1) into eq. (1.1) and multiply $u_V^*(i)\otimes u_V^*(j)$ from the left on both of the sides. Then we obtain the coupled equations

$$(E-m_V-\omega a^*a)\phi_{01}=ga\phi_{10}, \qquad (E-m_N-\omega a^*a)\phi_{10}=ga^*\phi_{01}. \tag{2.2}$$

We define operator G by the relation $(E-m_N-\omega a^*a)G=I$, then we can eliminate ϕ_{10} from (2.2) as follows:

$$(E-m_V-\omega a^*a-g^2aGa^*)\phi_{01}=0. \tag{2.3}$$

The oparator in (2.3) becomes diagonal when we use the usual representation $a=(\sqrt{n}\delta_{n+1,n'})$, and the n-th diagonal element, d_n say, is given by $d_n=E-m_V-(n-1)\omega-g^2n/(E-m_N-n\omega)$. From equation $d_n=0$, we obtain the eigenvalues $E=(1/2)(m_V+m_N)+(n-(1/2))\omega\pm(1/2)\sqrt{D}$, $n=1$, 2, 3, $\cdots$, where $D=[\omega-(m_V-m_N)]^2+4g^2n$. Conclusion for Case 1: Each of eigenvalues has two branch points in the g-plane, and they tend to zero as n tends to infinity. Hence, for a fixed value of g, we obtain at most a finite number of eigenvalues of eq. (1.1), by the use of perturbation method.

3. *Case* 2. We put $H^0=m_VV^*V+m_NN^*N+\sum_k\omega_ka_ka_k$, and $H'=L^{-3/2}\sum_k f(\omega_k)(2\omega_k)^{-1/2}(V^*Na_k+N^*Va_k^*)$, where $\omega_k=\sqrt{k^2+\mu^2}$, and $f(\omega_k)$ is a real cut-off function. Basis space $\mathfrak{H}(H^0)$ has the same expression as that given in **1** provided that $\mathfrak{H}_\theta$ is an incomplete direct product space [1] of sequence $\{\mathfrak{H}_k\}$, i.e., $\mathfrak{H}_\theta=\prod_k\otimes\mathfrak{H}_k$, where $\mathfrak{H}_k$ is the Hilbert space spanned by normalized eigenvectors $\phi_{k,\beta(k)}$, $\beta(k)=0,1,2,\cdots$ of $a_k^*a_k$. Eigenvalues and eigenvectors of $\sum_k\omega_ka_k^*a_k$ are respectively given by $E_\beta=\sum_k\beta(k)\omega_k$ and $\Phi_\beta=\prod_{k,\beta\in F}\otimes\phi_{k,\beta(k)}$, where notation $\beta\in F$ means that $\beta(k)$ is equal to zero except for a finite number of monenta k. Sequence $\{\Phi_\beta\}$ is a complete orthonormalized set of $\mathfrak{H}_\theta$.

When we use the same expression as (2.1) for Ψ, and follow the same process as in **2**, we arrive at the equation

$$O\phi_{10}\equiv\Big[\Big(E-m_V-\sum_k\omega_ka_k^*a_k\Big) -\frac{g^2}{L^3}\sum_k f(\omega_k)(2\omega_k)^{-1/2}\sum_{k'}f(\omega_{k'})(2\omega_{k'})^{-1/2}a_kG_Na_{k'}^*\Big]\phi_{10}=0, \tag{3.1}$$

where G_N is defined by the relation $(E-m_N-\sum_k\omega_ka_k^*a_k)G_N=I$, and has the expression $G_N=\sum_\beta\Phi_\beta\Phi_\beta^*/(E-m_N-E_\beta)$. Operator O has non-diagonal elements and eq. (3.1) can be solved only partially as shown below. We define $\Phi_0=\prod_k\otimes\phi_{k,0}$. Then, among matrix elements $\langle\Phi_\beta,O\Phi_0\rangle$, only $\langle\Phi_0,O\Phi_0\rangle$ is non-vanishing. Hence, some eigenvalues are obtained from equation $\langle\Phi_0,O\Phi_0\rangle=0$, which can be written in the form

$$-x+\delta=\frac{g^2}{L^3}\sum_k\frac{f^2(\omega_k)}{2\omega_k(\omega_k-x)}, \tag{3.2}$$

where $x=E-m_N$ and $\delta=m_V-m_N$. Eq. (3.2) is no other than the secular equation in the first sector of the Lee model [2]. When L tends to infinity, eq. (3.2) becomes

$$-x+\delta=w\cdot P\int_{\mu}^{\infty} f^2(\omega)\frac{\omega^2-\mu^2}{\omega-x}d\omega, \tag{3.3}$$

where P means the principal value, and $w=g^2/(2\pi)^2$.

We examine the analyticity of x in the neighbourhood of $w=0$ in cases of two simple cut-off functions $f(\omega)$.

(1) $f(\omega)=1$ in $(0,\Omega)$ and $=0$ in (Ω,∞), and $\mu=0$. We notice that the integral in (3.3) is continuous at $\mu=0$ for $f(\omega)$ used here. In this case eq. (3.3) is reduced to $-x+\delta=wG_0(x)$, where

$$G_0(x)=P\int_0^{\infty}\frac{\omega}{\omega-x}d\omega=\Omega+x\log\left|\frac{\Omega}{x}-1\right|.$$

Though the curve of $G_0(x)$ consists of three separate parts, the part given in $(0,\Omega)$ is necessary for our purpose, since $w\approx 0$. In this interval the above equation becomes $-x+\delta=w(\Omega+x\log(\omega/x-1))$, and it is written in the form $w=F(z)=-z/(\Omega+(z+\delta)\log(\omega/(z+\delta)-1))$, where $z=x-\delta$. The singular point of $F(z)$ which is nearest to $z=0$, is at $z=-\delta$, and the power series $F(z)=a_1z+a_2z^2+\cdots$ is convergent in circle $C:|z|<R(<\delta)$, and $a_1=1/(\Omega+\delta\log(\Omega/\delta-1))$. We put $M=\max_C|F(z)|$, then $M\cong R/\Omega$ when Ω is sufficiently large. According to a theorem on the inverse function [3], the power series of the inverse function of $w=F(z)$ has a radius of convergence not less than S defined by $S=|a_1|R+2M-2\sqrt{|a_1|RM+M^2}$. Since $S\to 0$ for $\Omega\to\infty$, we can conclude that the perturbation expansion of x in which δ is the zero-order approximation, converges when coupling constant g is sufficiently small. The domain guaranteed for the convergence becomes narrow indefinitly when Ω becomes large.

(2) $f^2(\omega)=e^{-p\omega}$, $p>0$ and $\mu=0$. The secular equation is given by $-x+\delta=wG(x)$, where $G(x)=P\int_0^{\infty}e^{-p\omega}\omega/(\omega-x)d\omega=p^{-1}-xe^{-px}\overline{\mathrm{Ei}}(px)$, $\overline{\mathrm{Ei}}(t)=\gamma+\log t+\sum_{n=1}^{\infty}t^n/n!\,n$ and $\gamma=0.577\cdots$ is the Euler constant [4]. We put $D(t)=e^{-t}\,\overline{\mathrm{Ei}}(t)$ and $z=x-\delta$, and write the secular equation in the form $w=F(z)=-pz/(1-p(z+\delta)\,\mathrm{D}(p(z+\delta)))$. Then it is seen that w tends to $p\delta$ when z tends to $-\delta$, and the inverse function of $w=F(z)$ is not regular at $w=p\delta$.

Proof. Suppose that z is regular at $w=p\delta$, put $p(z+\delta)=\sum_{n=1}^{\infty}c_n(w-p\delta)^n$ and insert it into the equation $w-p\delta=-p(z+\delta)(1-p\delta D(p(z+\delta)))/(1-p(z+\delta)\,\mathrm{D}(p(z+\delta)))$ which is derived from $w=F(z)$. Then the identity $1-\sum_1^{\infty}c_n(w-p\delta)^n\,\mathrm{D}(\sum_1^{\infty}c_n(w-p\delta)^n)=-\sum_1^{\infty}c_n(w-p\delta)^{n-1}(1-p\delta\,\mathrm{D}(\sum_1^{\infty}c_n(w-p\delta)^n))$ should hold valid. However, when w tends to $p\delta$, the right hand side tends to infinity or zero according as $c_1\neq 0$ or $=0$, respectively, while the left hand side always tends to unity for

$w \to p\delta$. Q.E.D.

Our conclusion is that the radius of convergence of the perturbation expansion tends to zero as p tends to zero.

4. *Case* 3. To the reaction (1.3) correspond the Hamiltonians $H^0 = m_V V^* V + m_N N^* N + \sum_k \omega_k a_k^* a_k$, $H' = L^{-3/2} \sum_k f(\omega_k)(2\omega_k)^{-1/2}(V^* N^* a_k + N V a_k^*)$. When we use expression (2.1) for Ψ in eq. (1.1), we obtain the coupled equations

$$\left(E - \sum_k \omega_k a_k^* a_k\right) \phi_{00} = g L^{-3/2} \sum_k f(\omega_k)(2\omega_k)^{-1/2} a_k^* \phi_{11}, \tag{4.1}$$

$$\left(E - m_V - m_N - \sum_k \omega_k a_k^* a_k\right) \phi_{11} = g L^{-3/2} \sum_k f(\omega_k)(2\omega_k)^{-1/2} a_k \phi_{00}. \tag{4.2}$$

In order to solve these equations, we define operator G by the relation $(E - \sum_k \omega_k a_k^* a_k) G = I$. Then we can write $G = \sum_\beta \Phi_\beta \Phi^*{}_\beta / (E - E_\beta)$. By using G, we obtain, from (4.1) and (4.2), the equation for ϕ_{11}:

$$\left[\left(E - m_V - m_N - \sum_k \omega_k a_k^* a_k\right) - g^2 L^{-3} \sum_k \frac{f(\omega_k)}{\sqrt{2\omega_k}} \sum_{k'} \frac{f(\omega_{k'})}{\sqrt{2\omega_{k'}}} a_k G a_{k'}^*\right] \phi_{11} = 0. \tag{4.3}$$

When we put $E + m_N = E'$ and $m_V + 2m_N = m_V'$, eq. (4.3) becomes $[(E' - m_V' - \sum_k \omega_k a_k^* a_k) - g^2 L^{-3} \sum_k f(\omega_k)(2\omega_k)^{-1/2} \sum_{k'} f(\omega_{k'})(2\omega_{k'})^{-1/2} a_k G a_{k'}^*] \phi_{11} = 0$, which has same form as (3.1), and G is expressed as $G = \sum_\beta \Phi_\beta \Phi^*{}_\beta / (E' - m_N - E_\beta)$. In this way, we arrive at the same equation as (3.3), where $x = E = E' - m_N$, $\delta = m_V + m_N = m_V' - m_N$, $x - \delta = E' - m_V' = E - m_V - m_N$. Accordingly, we get the same conclusions as in sect. 3.

5. Conclusion. In eigenvalue equation (1.1), we have examined the analyticity of eigenvalues in the neighbourhood of $g = 0$, regarding them as functions of coupling constant g. As simple models, we take three Cases 1, 2 and 3 given in the introduction. In Case 1, the problem can be solved completely, and our result is: Every eigenvalue has two branch points, and they as a whole accumulate at $g = 0$. In Cases 2 and 3, the problem can be solved partially. Instead of the usual method, we use the method of Green's function and arrive at the same secular equation as that derived in the first sector of the Lee model. The root of this secular equation which tends to an unperturbed eigenvalue as g tends to zero, can be expanded in a power series of g. The radius of convergence of this series tends to zero as the cut-off function $f(\omega)$ tends to unity. This circumstance may have some connections with the appearance of ghost states in the Lee model, because it occurs when $f(\omega)$ tends to unity.

References

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