67. *Generalized Prime Elements in a Compactly Generated l-Semigroup. II*

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(Comm. by Kenjiro SHODA, M. J. A., May 22, 1973)

Let L be a cl-semigroup with the conditions (1), (2), (3), (4) and $(*)$ in [2]. Moreover we impose that *the compact generator system Σ of L is closed under multiplication.* The main purpose of this note is to define principal φ-components of elements in L by using φ-primes in [2], and to prove that every element of L is decomposed into their principal φ-components.

3. Principal φ-Components.

Let a be an element of L, and u an element of Σ. The *(left) φ-residual* $a:u$ of a by u is defined to be the supremum of the set of all elements x with $\varphi(u)\varphi(x)\leq a$, $x\in\Sigma$. We suppose throughout this note that *there is such elements x for any $a\in L$ and any $u\in\Sigma$.* For a, b in L, the *(left) φ-residual* $a:b$ of a by b is defined as infimum of the $a:u$, where u runs over $\Sigma(b)$. Then we can prove the following properties: 1) $a\leq a'$ implies $a:b\leq a':b$, $b:a\geq b:a'$ and 2) $(\bigcap_{i=1}^n a_i):b=\bigcap_{i=1}^n(a_i:b)$ for $a, a', a_i, b\in L$.

Now it is not so evident that $a:b\geq a$ for a, b in L. To prove this, it is sufficient to show that $(a:u)\cup a=a:u$ for $a\in L$ and $u\in\Sigma(b)$. Take an arbitrary element x of $\Sigma((a:u)\cup a)$. Then we can choose an element y of $\Sigma(a:u)$ with $x\leq y\cup a$. Since $y\leq\sup\{x'\in\Sigma\mid\varphi(u)\varphi(x')\leq a\}$, we can find a finite number of compact elements $x_1,\cdots,x_n$ such that $y\leq\bigcup_{i=1}^n x_i$ and $\varphi(u)\varphi(x_i)\leq a$. Then we have $x\leq\bigcup_{i=1}^n x_i\cup a\leq\bigcup_{i=1}^n\varphi(x_i)\cup a$, $\varphi(x)\leq\bigcup_{i=1}^n\varphi(x_i)\cup a$, and $\varphi(u)\varphi(x)\leq\bigcup_{i=1}^n\varphi(u)\varphi(x_i)\cup\varphi(u)a\leq a$. Therefore we obtain $(a:u)\cup a\leq a$, $(a:u)\cup a=a$.

(3.1) Definition. Let p be a maximal φ-prime element belonging to an element a of L. The *principal φ-component of a by p*, denoted by $a(p)$, is the supremum of all $a:s$, s runs over $\Sigma'(p)$, if $p\neq e$. If $p=e$, $a(p)$ is defined to be a.

(3.2) Lemma. *$a\leq a(p)$ and $a(p)$ is φ-related to a for any maximal φ-prime element p belonging to a.*

Proof. If $p=e$, the assertion is trivial. So we suppose that $p\neq e$. We want to prove that $a(p)\cup a=a(p)$. For the sake of this, take an arbitrary element x of $\Sigma(a(p)\cup a)$. Then since there is an element y

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of $\Sigma(a(p))$ with $x \leq y \cup a$, we have $y \leq \sup_{s \in \Sigma'(p)} \{a : s\} = \sup_{s \in \Sigma'(p)} \{\sup [N_s]\} \leq \sup [\bigvee_{s \in \Sigma'(p)} N_s]$, where N_s is the set of the compact elements u with $\varphi(s)\varphi(u) \leq a$, and $\bigvee$ denotes the set-theoretical union. Then we can find a finite number of elements x_i of $\bigvee_{s \in \Sigma'(p)} N_s$ such that $y \leq \bigcup_{i=1}^{n} x_i$. Suppose that $x_i \in N_{s_i}$. Then we have $\varphi(s_i)\varphi(x_i) \leq a$, $x \leq y \cup a \leq \bigcup_{i=1}^{n} x_i \cup a \leq \bigcup_{i=1}^{n} \varphi(x_i) \cup a$, $\varphi(x) \leq \bigcup_{i=1}^{n} \varphi(x_i) \cup a$. Now let M^* be the kernel of $\Sigma'(p)$. Then we can find s_i^* of M^* such that $s_i^* \leq \varphi(s_i)$ for $i=1, 2, \cdots, n$. Take an element s^* of M^* such that $s^* \leq \prod_{i=1}^{n} s_i^*$. Then we have $\varphi(s^*) \leq \varphi(s_i)$, and $\varphi(s^*)\varphi(x) \leq \bigcup_{i=1}^{n} \varphi(s^*)\varphi(x_i) \cup a \leq \bigcup_{i=1}^{n} \varphi(s_i)\varphi(x_i) \cup a = a$. Hence $x \leq a : s^*$, $x \leq a(p)$. We get therefore $a(p) \cup a \leq a(p)$, $a(p) \cup a = a(p)$. Next, take an arbitrary element x of $\Sigma(a(p))$. Then we can show as above that $x' \leq \bigcup_{i=1}^{n} \varphi(x_i)$ for every x' in $\Sigma(\varphi(x))$, where $\varphi(s_i)\varphi(x_i) \leq a$. By using the above s^*, we obtain $\varphi(s^*)x' \leq \varphi(s^*)\varphi(x) \leq \bigcup_{i=1}^{n} \varphi(s^*)\varphi(x_i) \leq \bigcup_{i=1}^{n} \varphi(s_i)\varphi(x_i) \leq a$. If s is in $\Sigma'(p)$, we can show that $\varphi(s)$ is not less than a (by (2.5) in [2]). In particular, so is $\varphi(s^*)$. This means that x is φ-related to a. $a(p)$ is therefore φ-related to a.

By virture of the above proof we obtain the following:

(3.3) Corollary. *$a(p) = \sup_{s^* \in M^*} \{a : s^*\}$ for any kernel M^* of $\Sigma'(p)$.*

(3.4) Proposition. *Let p be any maximal φ-prime element belonging to an element a of L. Then $a(p)$ is less than every element b such that $b \geq a$ and every element of $\Sigma'(p)$ is φ-unrelated to b.*

Proof. Take an arbitrary element x of $\Sigma(a(p) \cup b)$. Then $x \leq x' \cup b$ for some x' of $\Sigma(a(p))$. Similar argument in the proof of (3.2) yields $x' \leq \bigcup_{i=1}^{n} \varphi(x_i) \cup b$, $\varphi(s^*)\varphi(x_i) \leq a \leq b$, where x_i and s^* are the same as in the proof of (3.2). Since s^* is φ-unrelated to b, there exists an element u of $\Sigma(\varphi(s^*))$ which satisfies that $uv \leq b$ implies $v \leq b$. Then we have $ux' \leq \bigcup_{i=1}^{n} \varphi(s^*)\varphi(x_i) \cup \varphi(s^*)b \leq b$. Hence $x' \leq b$, and hence $x \leq b$. Therefore we get $a(p) \cup b \leq b$, $a(p) \cup b = b$ and $a(p) \leq b$.

(3.5) Theorem. *Every element a of L is decomposed into the meet of all its principal φ-components.*

Proof. Let b be the element mentioned in (3.4). Let $\mathfrak{P}$ be the set of all maximal φ-prime elements belonging to a. Then by (3.2) we get $a \leq \inf_{p \in \mathfrak{P}} a(p)$. Conversely, if we take an arbitrary element x of $\Sigma(\inf_{p \in \mathfrak{P}} a(p))$, then $x \leq \sup_{s \in \Sigma'(p)} \{a : s\}$ for every $p \in \mathfrak{P}$. By the compactness we can take a finite number of elements $x_1, \cdots, x_n$ such that $x \leq \bigcup_{i=1}^{n} x_i$ with $\varphi(s_i)\varphi(x_i) \leq a$ for suitable $b_i \in \Sigma'(p)$. Then by the similar argument in the proof of (3.2), we have $\varphi(s_p^*)\varphi(x_i) \leq a$ for some s_p^* in $\Sigma'(p)$. Now we consider the set M_p of the elements u of Σ such that $\varphi(u)\varphi(x_i) \leq a$ for $i=1, \cdots, n$. (Existence of such an element u is easy to see.) Then M_p does not contained in $\Sigma(p)$ for each $p \in \mathfrak{P}$. By (2.6) in [2], u is φ-unrelated to a. Hence there exists an element u' of $\Sigma(\varphi(u))$ such that $u'v \leq a$ implies $v \leq a$. Then since $u'x_i' \leq \varphi(u)\varphi(x_i) \leq a$

for every x_i' in $\Sigma(\varphi(x_i))$, we have $x_i'\leq a, \varphi(x_i)\leq a$ for $i=1,\cdots,n$. Thus we get $x\leq\bigcup_{i=1}^n x_i\leq\bigcup_{i=1}^n \varphi(x_i)\leq a$. Therefore we obtain $\inf_{p\in\mathfrak{P}} a(p)\leq a$, completing the proof.

4. φ-Primary Decomposition.

(4.1) Definition. An element q of L is (left) *φ-primary,* iff whenever $\varphi(x)\varphi(y)\leq q$ implies $x\leq r_\varphi(q)$ or $y\leq q$ for $x,y\in\Sigma$.

φ-prime elements are evidently φ-primary elements.

Let M be a φ-system with kernel M^*. We suppose as in the case of [3] that *M meets $\Sigma(a)$ if and only if M^* meets $\Sigma(a)$ for every a in L.* (This condition holds for the trivial map $x\mapsto x, x\in\Sigma$.) Under the above condition we can prove that $r_\varphi(a\cap b)=r_\varphi(a)\cap r_\varphi(b)$ for any $a,b\in L$. The following properties are immediate by the definition of φ-radicals: 1) $a\leq b$ implies $r_\varphi(a)\leq r_\varphi(b)$ and 2) $r_\varphi(r_\varphi(a))=r_\varphi(a)$.

(4.2) Proposition. *If $q_1,\cdots,q_n$ is a finite number of φ-primary elements with the same φ-radicals, say $r_\varphi(q_i)=c$ for $i=1,\cdots,n$, then $q=q_1\cap\cdots\cap q_n$ is φ-primary and has the same radical c.*

The proof is the same as in [1].

(4.3) Proposition. *An element q is φ-primary if and only if $q:b=q$ for all elements b which are not less than $r_\varphi(q)$.*

Proof. Suppose that q is φ-primary and b is not less than $r_\varphi(q)$. Let y_0 be an element of $\Sigma(b)$ which is not less than $r_\varphi(q)$. If x is in $\Sigma(q:b)$, then there is a finite number of elements $x_1,\cdots,x_n$ such that $x\leq\bigcup_{i=1}^n x_i$ and $\varphi(y)\varphi(x_i)\leq q$ for all y in $\Sigma(b), i=1,\cdots,n$. We have in particular $\varphi(y_0)\varphi(x_i)\leq q$. Hence we get $x_i\leq q$ for $i=1,\cdots,n$ by the definition of φ-primarity. Therefore we have $x\leq q, q:b\leq q$ and $q:b=q$. Conversely, suppose that $\varphi(x)\varphi(y)\leq q$ and x is not less than $r_\varphi(q)$ for x,y in Σ. Then of course $\varphi(x)$ is not less than $\varphi(q)$. Hence we have $q:\varphi(x)=q$. Take an arbitrary element x' of $\Sigma(\varphi(x))$. Then we have $\varphi(x')\varphi(y)\leq q, y\leq q:x'$. Thus we get $y\leq\inf_{x'\in\Sigma(\varphi(x))}\{q:x'\}=q:\varphi(x)=q$. Therefore q is φ-primary. q.e.d.

A normal decomposition of elements with respect to φ-primary elements can be defined in the obvious way. If we suppose that *$q:q=e$ for every φ-primary element q* [3], we obtain an analogue of the uniqueness theorem of Lasker-Noether in commutative rings.

(4.4) Theorem. *Suppose that an element a has φ-primary decomposition. Then in any two normal decompositions of a, the number of φ-primary components as well as their φ-radicals are necessarily the same.*

The proof of this theorem is essentially the same as in [1].

References

[1] K. Murata: Primary decomposition of elements in compactly generated integral multiplicative lattices. Osaka J. Math., **7**, 97–115 (1970).
[2] K. Murata and Derbiau F. Hsu: Generalized prime elements in a compactly generated l-semigroup. I. Proc. Japan Acad., **49**, 134–139 (1973).
[3] K. Murata, Y. Kurata, and H. Marubayashi: A Generalization of prime ideals in rings. Osaka J. Math., **6**, 291–301 (1969).