66. *An Ergodic Theorem for a Semigroup of Linear Contractions*

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1. The purpose of the present paper is to extend a general ergodic theorem [1] and a general ergodic theorem of Abel type [4] in the discrete case to one in the continuous case.

2. Consider a σ-finite measure space $(X, \mathcal{F}, \mu)$ and also a measure space $(R^+, \mathcal{M}, dt)$ where $R^+=[0, \infty)$, $\mathcal{M}$ is the σ-algebra of all Lebesgue measurable subsets of R^+ and dt the Lebesgue measure on $\mathcal{M}$. Let L_1 be the real or complex Banach space of all equivalence classes of real or complex valued integrable functions on X.

Let $\{T_t : t \in R^+\}$ be a strongly continuous semigroup of linear contractions on L_1. Then it is known that, given $f \in L_1$, there exists a $\mathcal{M}\otimes\mathcal{F}$-measurable function g on $R^+\otimes X$ such that, for every t, $g(t, x) = (T_t f)(x)$ for a.a.x. Such a function g is uniquely determined up to a set of $dt\otimes d\mu$-measure zero. In what follows, $g(t, x)$ will be denoted by $(T_t f)(x)$. Then, by Fubini's theorem it is shown that, for a.a.x chosen suitably, $(T_t f)(x)$ is Lebesgue integrable on any bounded subinterval of R^+.

A family $\{p_t : t \in R^+\}$ of nonnegative measurable (not necessarily integrable) functions on X is called $\{T_t\}$-*admissible* if it satisfies

(i) *Admissibility.* $f \in L_1$ and $|f| \leqslant p_t$ for some t imply $|T_t f| \leqslant p_{s+t}$ for all s;

(ii) *Continuity.* There exists a strictly positive L_1-function p such that $\lim_{t\to s} \||p_t - p_s| \wedge p\| = 0$ for all s, where $q \wedge p$ means $\min(q, p)$.

Lemma 1. *Let $\{p_t : t \in R^+\}$ be $\{T_t\}$-admissible. Then there exists an $\mathcal{M}\otimes\mathcal{F}$-measurable function g on $R^+\otimes X$ such that, for every t, $g(t, x) = p_t(x)$ for* a.a.x. *Such a function g is uniquely determined up to a set of $dt\otimes d\mu$-measure zero.*

Proof. Define $p_{t,n}(x) = p_{[nt]/n}(x)$, where $[nt]$ is the integral part of nt. Then $p_{t,n}(x)$ is $\mathcal{M}\otimes\mathcal{F}$-measurable and, for every t,

$$\lim_{n\to\infty} \||p_{t,n} - p_t| \wedge p\| = 0.$$

On the other hand, since

$$|p_{t,m} - p_{t,n}| \wedge p \leqslant 2(|p_{t,m} - p_t| \wedge (p/2)) + 2(|p_{t,n} - p_t| \wedge (p/2)),$$

so

$$\lim_{m,n\to\infty} \||p_{t,m}-p_{t,n}|\wedge p\|=0.$$

Hence, by Fubini's theorem it is shown that, for any bounded subinterval $[a, b]$ of R^+,

$$\lim_{m,n\to\infty} \|||p_{t,m}-p_{t,n}|\wedge p\|| =0,$$

where $\||\cdot\||$ means the $L_1([a,b]\otimes X)$-norm. Recall now that p is strictly positive. Then it is shown that $\{p_{t,n}(x)\}_{n\geqslant 0}$ has a subsequence which converges to an $\mathcal{M}\otimes\mathcal{F}$-measurable function $h(t,x)$ a.e. in $R^+\otimes X$. Hence there exists a subset E of R^+ with dt-measure zero such that, for every $t\notin E$, $h(t,x)=p_t(x)$ for a.a.x. Define

$$g(t,x)=\begin{cases} h(t,x) & \text{if } t\notin E \\ p_t(x) & \text{if } t\in E. \end{cases}$$

Then it is easily seen that g is the desired.

In what follows, the function $g(t,x)$ in Lemma 1 will be denoted by $p_t(x)$.

We are now in a position to state our theorem.

Theorem. *Let $\{T_t : t\in R^+\}$ be a strongly continuous semigroup of linear contractions on L_1 and let $\{p_t : t\in R^+\}$ be $\{T_t\}$-admissible. Then, for every $f\in L_1$, two limits*

$$\lim_{\alpha\to\infty}\int_0^\alpha (T_tf)(x)dt\Big/\int_0^\alpha p_t(x)dt$$

and

$$\lim_{\lambda\downarrow 0}\int_0^\infty e^{-\lambda t}(T_tf)(x)dt\Big/\int_0^\infty e^{-\lambda t}p_t(x)dt$$

exist as finite values and coincide with each other a.e. on the set

$$\left\{x : \int_0^\infty p_t(x)dt>0\right\}.$$

Remark. When every p_t is integrable and (ii) is of the form: $\lim_{t\to s}\|p_t-p_s\|=0$, existence of

$$\lim_{\alpha\to\infty}\int_0^\alpha (T_tf)(x)dt\Big/\int_0^\alpha p_t(x)dt$$

is proved by Y. Kubokawa [3]. When every T_t is a positive linear contraction and $0\leqslant g\in L_1$, $p_t=T_tg$, existence of

$$\lim_{\lambda\downarrow 0}\int_0^\infty e^{-\lambda t}(T_tf)(x)dt\Big/\int_0^\infty e^{-\lambda t}(T_tg)(x)dt$$

is proved by D. A. Edwards [2].

3. We shall prove the theorem. For the proof we need some preparations.

Let T be a linear contraction on L_1. Then, a sequence $\{q_n\}_{n\geqslant 0}$ of nonnegative measurable functions on X is called *T-admissible* if $f\in L_1$ and $|f|\leqslant q_n$ for some n imply $|Tf|\leqslant q_{n+1}$.

Lemma 2 (R. V. Chacon [1] and R. Sato [4]). *Let T be a linear contraction on L_1 and let $\{q_n\}_{n\geqslant 0}$ be T-admissible. Then, for every $f\in L_1$,* *two limits* $\lim_{n\to\infty}\sum_{k=0}^{n}T^kf/\sum_{k=0}^{n}q_k$ *and* $\lim_{r\uparrow 1}\sum_{k=0}^{\infty}r^kT^kf/\sum_{k=0}^{\infty}r^kq_k$ *exist as finite values and coincide with each other a.e. on the set* $E=\{x:\sum_{k=0}^{\infty}q_k(x)>0\}$, *and also* $\lim_{n\to\infty}T^nf/\sum_{k=0}^{n-1}q_k=0$ *a.e. on* E. *Further,* $\lim_{n\to\infty}q_n/\sum_{k=0}^{n-1}q_k=0$ *at any point in* E *where*

$$\lim_{n\to\infty}\sum_{k=0}^{n}T^kf\Big/\sum_{k=0}^{n}q_k$$

exists as a finite value and does not vanish.

Lemma 3 (Y. Kubokawa [3]). *Under the same hypothesis as in the theorem, there exists a strongly continuous semigroup* $\{S_t:t\in R^+\}$ *of positive linear contractions on* L_1, *called the linear modulus of* $\{T_t:t\in R^+\}$, *such that* $|T_tf|\leqslant S_t|f|$ *for all* $f\in L_1$ *and* $\{p_t:t\in R^+\}$ *is* $\{S_t\}$-*admissible.*

Lemma 4. *Under the same hypothesis as in the theorem, let*

$$q_n(x)=\int_n^{n+1}p_t(x)dt\qquad(n=0,1,2,\cdots).$$

Then $\{q_n\}_{n\geqslant 0}$ *is* S_1-*admissible and so* T_1-*admissible, where* S_1 *is the operator in Lemma* 3.

Proof. It is enough to prove that $0\leqslant f\in L_1$ and $f\leqslant q_n$ imply $S_1f\leqslant q_{n+1}$. There exist a sequence $\{h_k\}_{k\geqslant 1}$ of nonnegative L_1-functions and a sequence $\{A_k\}_{k\geqslant 1}$ of measurable sets such that

$$\lim_{k\to\infty}\int_{X\setminus A_k}fd\mu=0,\qquad(1-1/k)1_{A_k}f\leqslant\int_n^{n+1}p_t\wedge h_kdt,$$

where 1_{A_k} is the indicator function of A_k. Then

$$(1-1/k)S_1(1_{A_k}f)\leqslant\int_n^{n+1}S_1(p_t\wedge h_k)dt\leqslant\int_n^{n+1}p_{t+1}dt=q_{n+1},$$
$$\lim_{k\to\infty}\|S_1f-(1-1/k)S_1(1_{A_k}f)\|=0,$$

so that $S_1f\leqslant q_{n+1}$.

Proof of the theorem. Let $\{S_t:t\in R^+\}$ be the linear modulus of $\{T_t:t\in R^+\}$. Define

$$g=\int_0^1T_tfdt,\quad h=\int_0^1S_1|f|dt,\quad q_n=\int_n^{n+1}p_tdt\quad(n=0,1,2,\cdots).$$

Then $g,h\in L_1$, $\int_k^{k+1}T_tfdt=T_1^kg$, $\int_k^{k+1}S_t|f|dt=S_1^kh$ $(k=0,1,2,\cdots)$, and $\{q_n\}_{n\geqslant 0}$ is S_1-admissible and so T_1-admissible by Lemma 4.

Now observe that if $n=[\alpha]$ and $r=e^{-\lambda}$ $(\lambda>0)$ then

$$\frac{\int_0^\alpha T_tfdt}{\int_0^\alpha p_tdt}=\left(\frac{\sum_{k=0}^{n-1}T_1^kg}{\sum_{k=0}^{n-1}q_k}+\frac{\int_n^\alpha T_tfdt}{\sum_{k=0}^{n-1}q_k}\right)\Bigg/\left(1+\frac{\int_n^\alpha p_tdt}{\sum_{k=0}^{n-1}q_k}\right),$$

$$\frac{\left|\int_n^\alpha T_tf\,dt\right|}{\sum_{k=0}^{n-1}q_k}\leqslant\frac{S_1^nh}{\sum_{k=0}^{n-1}q_k},\ \frac{\int_n^\alpha p_tdt}{\sum_{k=0}^{n-1}q_k}\leqslant\frac{q_n}{\sum_{k=0}^{n-1}q_k},$$

$$\frac{\int_0^\infty e^{-\lambda t} T_t f dt}{\int_0^\infty e^{-\lambda t} p_t dt} = \frac{\sum_{k=0}^\infty r^k T_1^k g + \left(\int_0^\infty r^t T_t f dt - \sum_{k=0}^\infty r^k T_1^k g\right)}{\sum_{k=0}^\infty r^k q_k + \left(\int_0^\infty r^t p_t dt - \sum_{k=0}^\infty r^k q_k\right)},$$

$$\left|\int_0^\infty r^t T_t f dt - \sum_{k=0}^\infty r^k T_1^k g\right| \leqslant (1-r) \sum_{k=0}^\infty r^k S_1^k h,$$

$$\left|\int_0^\infty r^t p_t dt - \sum_{k=0}^\infty r^k q_k\right| \leqslant (1-r) \sum_{k=0}^\infty r^k q_k.$$

Thus Lemma 2 completes the proof.

References

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