111. On the Characterization of the Linear Partial Differential Operators of Hyperbolic Type

By Kenji HORIE
Osaka University

(Comm. by Kenjiro SHODA, M. J. A., July 12, 1973)

§1. **Introduction.** In this note we shall consider a linear partial differential operator $P(D)$ of degree m with real constant coefficients in n variables. By α we denote multi-indices, that is, n-tuples $(\alpha_1, \cdots \alpha_n)$ of non-negative integers and by $|\alpha|$ their sum, that is $|\alpha|=\sum_{j=1}^n \alpha_j$. With $D_j=-\sqrt{-1}\,\partial/\partial x_j$, we set $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$. Then the symbol $P(D)$ represents a differential operator $P(D)=\sum_{|\alpha|\leqq m} a_\alpha D^\alpha$ and if $(\xi_1, \cdots, \xi_n)\in C^n$, then $P(\xi)$ does the polynomial $P(\xi)=\sum_{|\alpha|\leqq m} a_\alpha \xi^\alpha$, $\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$. This gives a one-to-one correspondence between polynomials and differential operators with constant coefficients. We shall call the operator $P(D)$ irreducible if the polynomial $P(\xi)$ is irreducible.

The aim of this note is to characterize the linear partial differential operator $P(D)$ by the support of the solution $u(x)\in C^\infty(R^n)$ of $P(D)u(x)=0$. If $u(x)$ satisfies $P(D)u(x)=0$, then $u(x)$ also satisfies $Q(D)P(D)u=0$ for arbitrary differential operator $Q(D)$. So we shall consider only irreducible linear partial differential operators.

Cohoon [1] proved the following theorem:

Theorem A. *There exists a nontrivial $u(x)$ in $C^\infty(R^n)$ such that $P(D)u(x)=0$ in R^n and such that the support of $u(x)$ is contained in $\{x\in R^n; |x_k|\leqq R$, for $k=1,2,\cdots n-1\}$ if and only if $P(D)$ is of the form*

$$P(D)=aD_n^m+\sum_{|\alpha|<m} b_\alpha D^\alpha$$

where $a\ (\neq 0)$ and $b_\alpha\ (|\alpha|<m)$ are real constants.

Then we ask when there exists a nontrivial $u(x)$ in $C^\infty(R^n)$ such that $P(D)u(x)=0$ in R^n and such that the support of $u(x)$ is contained in $\{x\in R^n; |x_k|\leqq R$ for $k=1,\cdots,n-2$ and $(r|x_n|+R)^2-x_{n-1}^2\geqq 0\}$ for $r\geqq 0$. It is the purpose of this note to answer this question.

The author thanks Professor S. Koizumi for his helpful discussions to the material of this note.

§2. **Definitions and theorem.** By $P_m(D)$ we shall denote the principal part of $P(D)$. According to Hörmander [3] the operator $P(D)$ is called hyperbolic with respect to $N\in R^n$, if $P_m(N)\neq 0$ and if there is a constant τ_0 such that $P(\xi+i\tau N)\neq 0$, when $\tau<\tau_0$ and $\xi\in R^n$. For the principal part $P_m(D)$ the definition of hyperbolicity is particularly simple by the following theorem.

Theorem. *The principal part $P_m(D)$ of $P(D)$ is hyperbolic with respect to N if and only if $P_m(N)\neq 0$ and the equation*

$$P_m(\xi+\tau N)=0$$

has only real roots when ξ is real.

If $P(D)$ is hyperbolic with respect to N, we shall denote by $\Gamma(P,N)$ the set of all real θ such that polynomial $P_m(\theta+\tau N)$ has only negative root τ. Then $\Gamma(P,N)$ is the component of N in the open set $\{\theta; P_m(\theta)\neq 0\}$ and is a convex cone with vertex at 0. By C^* we shall denote dual cone $\{x\in R^n; \langle x,\theta\rangle\geqq 0, \theta\in C\}$ of cone C.

Let e be the vector $(0,\cdots,0,1)\in R^n$. Let us introduce the domain $T_r=\{x\in R^n; |x_k|\leqq R, k=1,\cdots n-2 \text{ and } (r|x_n|+R)^2-x_{n-1}^2\geqq 0\}$, two cones $C_r=\{x\in R^n; x_n^2-(rx_{n-1})^2>0, x_n>0\}$, $C'_r=\{x\in R^n; x_n^2-(rx_{n-1})^2>0, x_n<0\}$ and the half space $H_N=\{x\in R^n; \langle x,N\rangle\geqq 0\}$.

We shall prove the following theorem.

Theorem. *Suppose $P(D)$ is an irreducible linear partial differential operator of degree m. Then there exists a nontrivial $u(x)$ in $C^\infty(R^n)$ such that* (i) *$P(D)u(x)=0$ in R^n,* (ii) *the support of $u(x)$ is contained in T_r if and only if $P(D)$ is of the form*

$$P(D)=a\prod_{i=1}^{m}(D_n+b_iD_{n-1})+\sum_{|\alpha|<m}c_\alpha D^\alpha,\qquad |b_i|\leqq r, \tag{1}$$

where $a(\neq 0)$, $b_i(i=1,\cdots,m)$ and c_α ($|\alpha|<m$) are real constants.

Theorem A is obtained by setting $r=0$ in this theorem. We show this theorem as a consequence of following two lemmas.

Lemma 1. *There exists a nontrivial $u(x)$ in $C^\infty(R^n)$ which satisfies* (i) *and* (ii) *if and only if the cone C_r is contained in $\Gamma(P_m,e)$.*

Lemma 2. *The cone C_r is contained in $\Gamma(P_m,e)$ if and only if $P(D)$ is of the form* (1).

§3. Proofs of Lemma 1 and Lemma 2. We first assume that $C_r\subset\Gamma(P_m,e)$. Let $\phi(x)$ be a C^∞ function of Gevrey class $\delta(1<\delta<m/m-1)$ with the support in $\{x\in R^n; |x_k|\leqq R, k=1,\cdots,n\}$. By the lemma 5.7.4 of Hörmander [3], there exists a function $U_k(\xi',x_n)$ which satisfies

$$P(\xi',D_n)U_k(\xi',x_n)=0,\qquad \xi'=(\xi_1,\cdots,\xi_{n-1}) \tag{2}$$

$$D_n^jU_k(\xi',0)=0,\quad \text{if } 0\leqq j,\ k<m \quad\text{and}\quad j\neq k, \tag{3}$$

$$D_n^kU_k(\xi',0)=1,\qquad \text{if } 0\leqq k<m, \tag{4}$$

and for some constant K

$$|D_n^lU_k(\xi',x_n)|\leqq K^{l+1}(|\xi'|+1)^{l+m-k}\exp[K|x_n|(|\xi'|+1)^{1-1/m}] \tag{5}$$

when $(\xi',x_n)\in R^n$ and $l=0,1,2,\cdots$.

Now let us consider

$$v(\xi',x_n)=\sum_{k=0}^{m-1}(D_n^k\hat{\phi}_n(\xi',0))U_k(\xi',x_n) \tag{6}$$

where $\hat{\phi}_n(\xi',x_n)=\int e^{-i\langle x',\xi'\rangle}\phi(x)dx'$. Using (3), (4), (5) and Paley-Wiener theorem, it follows that

(7) $|D_n^j v(\xi', x_n)| \leqq \sum_{k=0}^{m-1} |D_n^k \hat{\phi}_n(\xi', 0)| \cdot |D_n U_k(\xi', x_n)|$

$$\leqq \sum_{k=0}^{m-1} K_B K^{j+1}(|\xi'|+1)^{j+m-k} \exp[K|x_n|(|\xi'|+1)^{1-1/m} B|\xi'|^{1-1/m}]$$

$$\leqq CK_B K^{j+1}(|\xi'|+1)^{j+m} \exp[(K|x_n|-B)|\xi'|^{1-1/m}]$$

for some constant C and $B \geqq R$. In particular this shows that $v(\xi', x_n)$ is in $L_1(R^{n-1}_{\xi'})$ for fixed x_n. We can set $u(x)=\mathcal{F}_n^{-1}[v(\xi', x_n)]$ where $\mathcal{F}_n^{-1}$ is a partial inverse Fourier transform with respect to $\xi_1, \cdots, \xi_{n-1}$. Since B can be chosen arbitrary large, from (7) it follows that

(8) $||u(x', x_n)||_{2,k_s} = ||(1+|\xi'|^2)^{s/2} v(\xi', x_n)||_2$

$$\leqq (cK_B K)^2 \sum_{|\alpha'| \leqq s} \int |\xi'|^{2(|\alpha'|+m)} \exp[2(K|x_n|-B)|\xi'|^{1-1/m}] d\xi' < \infty.$$

Since s can be chosen arbitrary large, from (8) and Sovolev's lemma, we have

$$u(x', x_n) \in C^\infty(R^{n-1}_{x'}).$$

From this and (5), it follows that $u(x) \in C^\infty(R^n)$.
Furthermore, from (2), (3), (4) and (6) we have

(9) $$P(D)u(x)=0, \quad \text{in } R^n$$

and

(10) $$D_n^j u(x', 0) = D_n^j \phi(x', 0), \quad 0 \leqq j < m.$$

Since $\operatorname{supp} \phi(x', 0) \subset \{x \in R^n;\ |x_k| \leqq R, k=1, \cdots, n-1 \text{ and } x_n=0\}$, if we apply Corollary 5.3.2 of Hörmander [3], we can obtain,

$$\operatorname{supp} U \cap H_e \subset \{x \in R^n;\ |x_k| \leqq R, k=1, \cdots, n-1 \text{ and } x_n=0\} + \Gamma(P_m, e)^*.$$

Similarly we have

$$\operatorname{supp} U \cap H_{(-e)} \subset \{x \in R^n;\ |x_k| \leqq R, k=1, \cdots, n-1, x_n=0\} + \Gamma(P_m, -e)^*.$$

Since $\Gamma(P_m, e)^* \subset C_r^*$ and $\Gamma(P_m, -e)^* \subset C_r'^*$, we have $\operatorname{supp} U \subset T_r$.

To prove the converse we consider the hyperplane $\Sigma(N)=\{x \in R^n;\ \langle x, N\rangle=0\}$, where N is a vector in C_r. It is obvious that $\Sigma(N) \cap T_r$ is compact. Then we have $N \in \Gamma(P_m, e)$. Because by the theorem of John [2], unless $N \in \Gamma(P_m, e)$, u vanishes identically in $\Sigma(N)$ and by translations, it follows that u vanishes identically in R^n, which contradicts the assumption. This completes the proof of Lemma 1.

Proof of Lemma 2. We first assume that $C_r \subset \Gamma(P_m, e)$. Let N be the vector such that $N=(0, \cdots, N_{n-1}, N_n) \in C_r$. Let us consider the following equation with respect to ζ.

(11) $$P_m(\xi_1, \xi_2, \cdots, \xi_{n-2}, \zeta N_{n-1}, \zeta N_n)=0.$$

Suppose that for some $(\xi_1, \cdots, \xi_{n-2}, 0, 0) \in R^n$ we could find nonzero complex number ζ which satisfies (11). But Theorem 5.5.3 of Hörmander [3] tells us that ζ must have been real.
Then we have

(12) $$(\xi_1\zeta^{-1}, \xi_2\zeta^{-1}, \cdots, \xi_{n-2}\zeta^{-1}, N_{n-1}, N_n) \in C_r.$$

From this and the assumption we conclude that

$$(\xi_1, \xi_2, \cdots, \xi_{n-2}, \zeta N_{n-1}, \zeta N_n)$$

is a hyperbolic direction of $P_m(D)$ and consequently that

(13) $$P_m(\xi_1, \cdots, \xi_{n-2}, \zeta N_{n-1}, \zeta N_n) \neq 0.$$

This contradicts that ζ is a root of equation of (11). Thus it is proved that the equation

(14) $$P_m(\xi_1, \cdots, \xi_{n-2}, \zeta N_{n-1}, \zeta N_n) = 0$$

has only $\zeta=0$ as a root. Furthermore

$$P_m(\xi_1, \cdots, \xi_{n-2}, \zeta N_{n-1}, \zeta N_n) = \sum_{|\alpha|+\beta+\gamma=m} a_{\alpha\beta\gamma}\xi''^{\alpha}(\zeta N_{n-1})^{\beta}(\zeta N_n)^{\gamma}$$
$$= \sum_{k=0}^{m-1} \Bigg(\sum_{\substack{|\alpha|=m-k \\ \beta+\gamma=k}} \xi''^{\alpha} N_{n-1}^{\beta} N_n^{\gamma} \Bigg) \zeta^k + \Bigg(\sum_{\beta+\gamma=m} a_{0\beta\gamma} N_{n-1}^{\beta} N_n^{\gamma} \Bigg) \zeta^m.$$

We have

(15) $$\sum_{\substack{|\alpha|=m-k \\ \beta+\gamma=k}} a_{\alpha\beta\gamma}\xi''^{\alpha} N_{n-1}^{\beta} N_n^{\gamma} = 0$$

where $k=0, \cdots, m-1, (0, \cdots, 0, N_{n-1}, N_n) \in C_r$.

Let us set $\eta = N_{n-1}N_n^{-1}$ by $N_n \neq 0$. Since C_r is a cone, we have

(16) $$\sum_{\beta=0}^{k} \Bigg(\sum_{\substack{|\alpha|=m-k \\ \gamma=k-\beta}} a_{\alpha\beta\gamma}\xi''^{\alpha} \Bigg) \eta^{\beta} = 0$$

for all ξ'' in R^{n-2} and η in $(-r^{-1}, r^{-1})$. From this we conclude that $a_{\alpha\beta\gamma}=0$ for all α in N^{n-2} with $|\alpha| = m-(\beta+\gamma)$ for all (β, γ) with $0 \leqq \beta+\gamma \leqq m-1$ and $\beta \geqq 0, \gamma \geqq 0$. Thus $P_m(\xi) = Q(\xi_{n-1}, \xi_n)$ for some suitable homogeneous polynomial of degree m in two variables of ξ_{n-1} and ξ_n. Then by the fundamental theorem of algebra, we can find the complex numbers a and $b_i (i=1, \cdots, m)$ such that,

(17) $$P_m(\xi) = a \prod_{i=1}^{m} (\xi_n + b_i \xi_{n-1}), \text{ where } a \neq 0.$$

Since e is a hyperbolic direction of $P_m(D)$, the $b_i (i=1, \cdots, m)$ are real constants. Let c and d be $\mathrm{Max}\{b_i ; b_i \geqq 0\}$, $\mathrm{Min}\{b_i ; b_i \leqq 0\}$, respectively. Then we have

(18) $$\Gamma(P_m, e) = \{x \in R^n ; x_n + cx_{n-1} > 0, x_n + dx_{n-1} > 0\}.$$

By the assumptions, it follows that $c \leqq r, d \geqq -r$. Thus, $|b_i| \leqq r$, for $i=1, \cdots m$.

Conversely, if $P(D)$ is of the form (1) then using (18), we conclude that $C_r \subset \Gamma(P_m, e)$.

The proof of Lemma 2 is complete.

References

[1] D. K. Cohoon: A characterization of the linear partial differential operato $P(D)$ which admit a nontrival C^∞ solutions with support in an open prism with bounded cross section. J. Differential Equations, **8**, 195–201 (1970).

[2] F. John: Non-admissible data for differential equations with constant coefficients. Comm. Pure. Appl. Math., **10**, 391–398 (1957).

[3] Lars Hörmander: Linear Partial Differential Operators. Springer-Verlag, New York, Berlin (1963).