143. Theorems on the Finite-dimensionality of Cohomology Groups. IV

By Takahiro KAWAI
Research Institute for Mathematical Sciences, Kyoto University

(Comm. by Kôsaku YOSIDA, M. J. A., Nov. 12, 1973)

In this note we discuss theorems related to the finite-dimensionality of cohomology groups attached to a differential complex. The main purpose of § 2 is to derive the vanishing of the cohomology groups from their finite-dimensionality by deforming the boundary suitably. Note that the result in § 1 is rather isolated from the main stream of the results in this series of our notes in its nature, since the scope of its applicability is completely restricted to the system of linear differential equations with constant coefficients. We have included it here because of its importance. The details and complete arguments will be given somewhere else.

§ 1. In this section we treat exclusively the system of linear differential equations with constant coefficients. First let us recall the following notion of k-convexity (of an open set in $\boldsymbol{R}^n$) due to Palamodov [5].

Definition (Palamodov [5] Part II § 11). An open set Ω in $\boldsymbol{R}^n$ is said to be completely k-convex if it satisfies the following condition:

There exists $h(x)$ in $C^2(\Omega)$ such that its Hessian matrix Hess $h(x)$ $=\{\partial^2 h/\partial x_i \partial x_j\}_{1\leq i,j\leq n}$ has at least k positive eigenvalues and that K_c $=\{x\in\Omega\,;\, h(x)\leqq c\}$ is compact for any $c\in\boldsymbol{R}$.

Remark. The notion of complete k-convexity of Ω introduced above has nothing to do with that of q-convexity of the system of pseudo-differential equations introduced in Sato-Kawai-Kashiwara [6] Chapter III § 2.3, though both notions have stemmed from the notion of q-convexity employed in the theory of analytic functions of several complex variables. In § 2 we use the notion of q-convexity of the system of pseudo-differential equations.

Theorem 1. *Let $\mathcal{M}$ be a system of linear differential equations of finite order with* ***constant*** *coefficients. Take a completely $(n-k)$-convex open set Ω in $\boldsymbol{R}^n$. Then*

$$\mathrm{Ext}^j_{\mathcal{D}}(\Omega\,;\mathcal{M},\mathcal{B})=0 \qquad \text{for } j>k \tag{1}$$

holds. Here $\mathcal{B}$ and $\mathcal{D}$ denote the sheaf of hyperfunctions and linear differential operators, respectively.

This result is a generalization of the result of Komatsu concerning the vanishing of cohomology groups on a convex open set having the hyperfunction solution sheaf of the system $\mathcal{M}$ as their coefficients. (Komatsu [4] Theorem 3.) It seems that the result corresponding to Theorem 1 in the case of distribution solutions or C^∞-solutions is not known in such a full generality. (See Palamodov [5] Part II § 11 Corollary 2.) The result for real analytic solutions will be given somewhere else.

§ 2. In this section we treat the elliptic system $\mathcal{M}$ of linear differential equations defined on a paracompact and oriented real analytic manifold M. For the sake of simplicity we always assume that $\mathcal{M}$ has a free resolution by the sheaf $\mathcal{D}$ of linear differential operators. (See the remark in Kawai [2] p. 80.) We begin our discussions by the following Lemma 2, which is of its own interest. In the sequel we use the same notations as in Kawai [3] and do not repeat the definition of the "positive" tangential system $\mathcal{N}_+$ (or $\mathcal{N}_{j,+}$) of pseudo-differential equations defined via the "positive characteristics" S.S. $\mathcal{M}\cap G_+$ (or S.S. $\mathcal{M}\cap G_{j,+}$). As to their definitions we refer to Kawai [3].

Lemma 2. *Let Ω and Ω_j $(j=1,2,\cdots)$ be relatively compact open sets with C^∞ boundary in M such that $\{\Omega_j\}_{j=1}^\infty$ constitute the fundamental system of neighbourhoods of the closure $\bar{\Omega}$ of Ω and that all the "positive" tangential systems $\mathcal{N}_+$ and $\mathcal{N}_{j,+}$ are $(q+1)$-convex or $(q-1)$-concave on their respective real characteristic variety. Then we can find some j_0 such that*

$$(2)\qquad \mathrm{Ext}^q_{\mathcal{D}}(\Omega_{j_0};\mathcal{M},\mathcal{B})\longrightarrow \mathrm{Ext}^q_{\mathcal{D}}(\Omega;\mathcal{M},\mathcal{B})\longrightarrow 0$$

holds.

In proving this lemma we use Theorem 1 in Kawai [3] and Théorème A in Grothendieck [1] p. 16.

By the aid of this lemma we can argue in an analogous way to Palamodov [5] Part II § 11 and obtain the following theorems.

Theorem 3. *Let Ω be a connected open set in M. Assume that there exists a real valued real analytic function $\varphi(x)$ defined on Ω which satisfies the following:*

(3) *$\Omega_t=\{x\in\Omega;\varphi(x)<t\}$ is relatively compact for any $t\in\boldsymbol{R}$.*

(4) *Ω_t has C^∞-boundary $\partial\Omega_t$ as long as $\Omega_t\neq\emptyset$.*

Assume further that the "positive" tangential system $\mathcal{N}_{t,+}$ is q-convex on its real characteristic variety (if $\partial\Omega_t\neq\emptyset$). Then we have the following isomorphism

$$(5)\qquad \mathrm{Ext}^j_{\mathcal{D}}(\Omega;\mathcal{M},\mathcal{B})\xrightarrow{\sim}\mathrm{Ext}^j_{\mathcal{D}}(\Omega_t;\mathcal{M},\mathcal{B})$$

if $j<q$ and $\Omega_t\neq\emptyset$.

Theorem 4. *Let Ω and φ be the same as in Theorem 3. Assume that $\mathcal{N}_{t,+}$ is q-concave on its real characteristic variety. Then we have the following isomorphism*

(6) $$\mathrm{Ext}^j_{\mathcal{D}}(\Omega\,;\mathcal{M},\mathcal{B})\xrightarrow{\sim}\mathrm{Ext}\quad(\Omega_t\,;\mathcal{M},\mathcal{B})$$
if $j>q+1$ *and* $\Omega_t\neq\emptyset$.

We have the following results as an obvious consequence of the above theorems.

Corollary 1. *Assume that* $K_0=\{x\in\Omega\,;\varphi(x)\leqq 0\}$ *reduces to a single point* x_0 *and that* $\mathcal{E}xt^j_{\mathcal{D}}(\mathcal{M},\mathcal{A})=0$ *holds near* x_0 *for* $1\leqq j<q$. *Here* $\mathcal{A}$ *denotes the sheaf of real analytic functions. Then the same hypotheses as in Theorem* 3 *imply that*

(7) $$\mathrm{Ext}^j_{\mathcal{D}}(\Omega\,;\mathcal{M},\mathcal{B})=0$$
holds if $1\leqq j<q$.

Corollary 2. *Assume that* K_0 *reduces to a single point* x_0 *and that* $\mathcal{E}xt^j_{\mathcal{D}}(\mathcal{M},\mathcal{A})=0$ $(j\geqq 1)$ *holds near* x_0. *Then the same hypotheses as in Theorem* 4 *imply that*

(8) $$\mathrm{Ext}^j_{\mathcal{D}}(\Omega\,;\mathcal{M},\mathcal{B})=0$$
holds if $j>q+1$.

By the analogous argument we can obtain the following semi-global result.

Theorem 5. *Let* Ω *and* φ *be the same as in Theorem* 3. (*It is sufficient to assume* (3) *only for* $t<1$ *in this case.*) *Assume that* K_0 *reduces to a point* x_0 *and that* $\mathcal{E}xt^j_{\mathcal{D}}(\mathcal{M},\mathcal{A})=0$ $(1\leqq j\leqq q)$ *holds near* x_0. *Assume further that* $\mathcal{N}_{t,+}$ *is either* $(q+1)$-*convex or* $(q-1)$-*concave on its real characteristic variety if* $0<t\ll 1$. *Then there exists* $t_0>0$ *such that*

(9) $$\mathrm{Ext}^q_{\mathcal{D}}(\Omega_{t_0}\,;\mathcal{M},\mathcal{B})=0$$
holds.

Erratum in Kawai [3]. Theorem 2 in Kawai [3] should be replaced by Corollary 2 and Theorem 5 above. (See also the remark below.) The assumption that $\mathcal{E}xt^j_{\mathcal{D}}(\mathcal{M},\mathcal{A})=0$ $(j\geqq 1)$ is clearly satisfied if $\mathcal{M}$ is with constant coefficients, which is the case treated in Theorem 2 in Kawai [3].

Remark. If $\mathcal{M}$ is with constant coefficients and if $\Omega\cap\boldsymbol{R}^n$ is convex, then Theorem 4 can be obviously used to assert the vanishing of $\mathrm{Ext}^j_{\mathcal{D}}(\Omega_t\,;\mathcal{M},\mathcal{B})$. In fact $\mathrm{Ext}^j_{\mathcal{D}}(\Omega\,;\mathcal{M},\mathcal{B})=0$ $(j\geqq 1)$ holds by a result of Komatsu [4]. Moreover the situation is the same in the case treated by Theorem 5, that is, we can relax the condition in Theorem 5 so that K_0 is a compact convex set.

References

[1] Grothendieck, A.: Produits tensoriels topologiques et espaces nucléaires. Mem. A. M. S. No. 16, A. M. S., Providence (1955).

[2] Kawai, T.: Finite-dimensionality of cohomology groups attached to systems of linear differential equations. J. Math. Kyoto Univ., **13**, 73–95 (1973).

[3] Kawai, T.: Theorems on the finite-dimensionality of cohomology groups. III. Proc. Japan Acad., **49**, 243–246 (1973).

[4] Komatsu, H.: Resolution by hyperfunctions of sheaves of solutions of differential equations with constant coefficients. Math. Annalen, **176**, 77–86 (1968).

[5] Palamodov, V. P.: Linear Differential Operators with Constant Coefficients. Nauka, Moscow (1967) (in Russian). (Translated into English (Springer, 1970) and into Japanese (Yoshioka, 1973)).

[6] Sato, M., T. Kawai, and M. Kashiwara: Microfunctions and pseudo-differential equations. Lecture Note in Mathematics No. 287. Springer, Berlin-Heidelberg-New York, pp. 265–529 (1973).