11. Note on Some Whitehead Products

By Yasutoshi NOMURA
College of General Education, Osaka University

(Comm. by Kenjiro SHODA, M. J. A., Jan. 12, 1974)

1. Introduction. For standard generators $\theta \in \pi_q(S^n)$ the problem whether Whitehead products $[\theta, \iota_n]$ are 0 or not has been investigated by various authors [1], [2], [7], [8]. In this note we are concerned with the question whether $[\theta, \iota_n] \in \theta_* \pi_{n+q-1}(S^q)$ or not. Following the Toda notation [13] our main result is stated as follows.

Theorem. *$[\theta, \iota_n]$ does not lie in the image of $\theta_* : \pi_{n+q-1}(S^q) \to \pi_{n+q-1}(S^n)$ for the following θ:*

η_n, $n \equiv 0, 1 \bmod 4$ *and* $n \geqq 5$; η_n^2, $n \equiv 0 \bmod 4$; ν_n, $n \equiv 1, 3 \bmod 8$ *and* $n \geqq 9$ or $n \equiv 0 \bmod 2$ *and* $n \geqq 6$; ν_n^2, $n \equiv 2 \bmod 4$ *and* $n \geqq 6$; σ_n, $n \equiv 1 \bmod 4$ *and* $n \geqq 13$ or $n \equiv 0 \bmod 2$ *and* $n \geqq 10$; $8\sigma_n$, $n \equiv 2 \bmod 4$ *and* $n \geqq 10$; ε_n, $n \equiv 1 \bmod 4$ *and* $n \geqq 13$; $\bar{\nu}_n$, $n \equiv 1 \bmod 4$ *and* $n \geqq 13$; μ_n, $n \equiv 1 \bmod 4$ *and* $n \geqq 13$; ρ_n, $n \equiv 1 \bmod 4$ *and* $n \geqq 21$; κ_n, $n \equiv 1 \bmod 4$ *and* $n \geqq 21$; ω_n, $n \equiv 1 \bmod 4$ *and* $n \geqq 21$; $\bar{\mu}_n$, $n \equiv 1 \bmod 4$ *and* $n \geqq 21$; ζ_n, $n \equiv 0 \bmod 2$ *and* $n \geqq 6$; $\bar{\kappa}_n$, $n \equiv 1 \bmod 4$ *and* $n \geqq 25$ or $n \equiv 0 \bmod 2$ *and* $n \geqq 8$; $\bar{\zeta}_n$, $n \equiv 0 \bmod 2$ *and* $n \geqq 6$; ν_n^*, $n \equiv 0 \bmod 2$ *and* $n \geqq 18$; $\eta_n \sigma_{n+1}$, $n \equiv 0, 1 \bmod 4$ *and* $n \geqq 12$; $\eta_n \mu_{n+1}$, $n \equiv 0 \bmod 4$ *and* $n \geqq 12$; $\eta_n \rho_{n+1}$, $n \equiv 0, 1 \bmod 4$ *and* $n \geqq 20$; $\eta_n \eta_{n+1}^*$, $n \equiv 0 \bmod 4$ *and* $n \geqq 24$; $\eta_n \bar{\mu}_{n+1}$, $n \equiv 0 \bmod 4$ *and* $n \geqq 24$.

Consequently, from a theorem of James [4] we may deduce

Corollary. *There exist no Poincaré complexes of the form $(S^n \bigcup_\theta e^{q+1}) \cup e^{n+q+1}$, where θ are elements exhibited in Theorem.*

2. Special cases of Toda's propositions. Some of the following lemmas are obtained as corollaries of Propositions 11.10 and 11.11 of Toda [13], but proofs may be given which are based on the results of James [3], Kervaire [6] and Paechter [12].

Lemma 2.1. *For $n \equiv 0 \bmod 4$, $n \geqq 4$, there exists $\tau_{n-1} \in \pi_{2n-1}(S^{n-1})$ such that $E\tau_{n-1} = [\eta_n, \iota_n]$ and $H(\tau_{n-1}) = \eta_{2n-3}^2$.*

Remark. This is obtained from Proposition 11.10, i) of [13] for $\alpha = \eta_{2n-4}$. According to [13], [10] we may take $\tau_3 = \nu' \eta_6$, $\tau_7 = \sigma' \eta_{14}$, $\tau_{11} = \theta'$, $\tau_{15} \equiv \eta^{*\prime} \bmod E\pi_{30}(S^{14})$ and $\tau_{19} = \bar{\beta}$.

Proof. Introduce the diagram

$$
\begin{array}{ccccccc}
 & & \pi_{2n-2}(S^{n-2}) & = & \pi_{2n-2}(S^{n-2}) & & \\
 & & \downarrow E & & \downarrow E^3 & & \\
\pi_n(R_{n-1}) & \xrightarrow{J} & \pi_{2n-1}(S^{n-1}) & \xrightarrow{E^2} & \pi_{2n+1}(S^{n+1}) & & \\
\uparrow \partial_1 & \searrow k & \downarrow H_1 & & \downarrow H_3 & & \\
\pi_{n+1}(R_{n+1}, R_{n-1}) \xrightarrow[\partial_2]{} & & \pi_n(R_{n-1}, R_{n-2}) & \xrightarrow{i} & \pi_n(R_{n+1}, R_{n-2}) & \xrightarrow{j} & \pi_n(R_{n+1}, R_{n-1}) \\
 & & \| & & \| & & \| \\
 & & \pi_n(S^{n-2}) & & \pi_n(V_{n+1,3})=Z_2 & & \pi_n(V_{n+1,2})=Z_2
\end{array}
$$

$$
\begin{array}{ccccc}
\to\pi_{n-1}(S^{n-2}) & \to & \pi_{n-1}(V_{n+1,3}) & \to & \pi_{n-1}(V_{n+1,2}) \\
\| & & \| & & \| \\
Z_2 \text{ or } Z & & Z_4 \text{ or } Z & & Z_2 \text{ or } 0
\end{array}
$$

in which row and columns are exact and the diagram commutes up to sign by James [3]. The values of homotopy groups of Stiefel manifolds are taken from Paechter [12]. We see that j is bijective, hence we may find $t\in\pi_{n+1}(R_{n+1}, R_{n-1})$ with $\partial_2 t=\eta^2_{n-2}$. Since $H_3E^2J\partial_1 t=0$, there is a $t'\in\pi_{2n-2}(S^{n-2})$ with $E^3t'=E^2J\partial_1 t$. Since $E:\pi_{2n-1}(S^{n-1})\to\pi_{2n}(S^n)$ is monic by $[\eta^2_{n-1}, \iota_{n-1}]=0$, it follows that $\tau_{n-1}=J\partial_1 t-Et'$ is what we wanted.

We now see from the well known information of vector fields on spheres that, if we write $n+1=m\cdot 2^c\cdot 16^d$ where m is odd$\geqq 3$ and $0\leqq c\leqq 3$ then there exists $\tau^{(\rho-1)}_{n-\rho+1}\in\pi_{2n-\rho}(S^{n-\rho+1})$, $\rho=2^c+8d$, such that $[\iota_n, \iota_n]=E^{\rho-1}\tau^{(\rho-1)}_{n-\rho+1}$ and $H(\tau^{(\rho-1)}_{n-\rho+1})\neq 0$ in $\pi_{2n-\rho}(S^{2n-2\rho+1})$. Special cases of this fact are needed in the sequel.

Consider the bundles $U_{n+1}\to S^{2n+1}$ and $Sp_{n+1}\to S^{4n+3}$ with characteristic classes $\gamma'_{2n}\in\pi_{2n}(U_n)$, $\gamma''_{4n+2}\in\pi_{4n+2}(Sp_n)$. In the light of the results of Ôguchi [11] and James-Whitehead [5] we may take for $\tau^{(1)}_{2n}\in\pi_{4n}(S^{2n})$ and $\tau^{(3)}_{4n}\in\pi_{8n+2}(S^{4n})$ (n: even) the images under Hopf-Whitehead homomorphisms.

Lemma 2.2. *For n even, $\tau^{(1)}_{2n}$ is of order 2 and $E\tau^{(1)}_{2n}=[\iota_{2n+1}, \iota_{2n+1}]$, $H(\tau^{(1)}_{2n})=\eta_{4n-1}$.*

Remark. This lemma is related to Proposition 11.10, ii) of [13] with $\alpha=\iota_{4n-2}$. According to [13, 10], we may take $\tau^{(1)}_4=\nu_4\eta_7$, $\tau^{(1)}_8=\sigma_8\eta_{15}+\bar{\nu}_8+\varepsilon_8$, $\tau^{(1)}_{12}=\theta$, $\tau^{(1)}_{16}\equiv\eta^*_{16}+\omega_{16}$ mod $\sigma_{16}\mu_{23}$, $\tau^{(1)}_{20}=\bar{\bar{\beta}}$. Note that $\pi_{2n}(R_{2n})=(Z_2)^3$ or $(Z_2)^2$ by [6].

Lemma 2.3. *For n even, $E^3\tau^{(3)}_{4n}=[\iota_{4n+3}, \iota_{4n+3}]$ and $H(\tau^{(3)}_{4n})=r\nu_{8n-1}$, where $r=\pm 1, \pm 3$.*

Remark. According to [13], [9] we may take $\tau^{(3)}_8=\sigma_8\nu_{15}$, $\tau^{(3)}_{16}=\nu^*_{16}+\xi_{16}$, $[\iota_{23}, \iota_{23}]=E^3\sigma^*_{20}$.

Lemma 2.4. *For $n\equiv 0$ mod 4, $n\geqq 4$, $\pi_{2n+3}(U_{n-1})\cong\pi_{2n+3}(U_n)$, $\pi_{2n+3}(U_n)$ is cyclic with generator $\gamma'_{2n}\nu_{2n}$ and $\pi_{2n+3}(U_{n-1})$ is generated by u^{n-1}_{2n+3} whose image under the J-homomorphism is denoted by $\bar{\tau}_{2n-2}\in\pi_{4n+1}(S^{2n-2})$. Then $E^3\bar{\tau}_{2n-2}=[\nu_{2n+1}, \iota_{2n+1}]$ and $H(\bar{\tau}_{2n-2})=\nu^2_{4n-5}$.*

3. Proof of Theorem.

Proposition 3.1. *Suppose that* $\beta\in\pi_q(S^{4k+1})$ *satisfies* $\eta_{8k-3}^2(E^{4k-2}\beta)\neq 0$, *where* $q\leqq 8k-4$. *If* $E^2:\pi_{q+4k-3}(S^{q-2})\to\pi_{q+4k-1}(S^q)$ *is epic (e.g.,* $q\geqq 4k+2$), *then* $[\eta_{4k}\beta,\iota_{4k}]\notin\eta_{4k}\beta\pi_{q+4k-1}(S^q)$.

Proof. We may write $\beta=E^2\beta'$. By Lemma 2.1, $[\eta_{4k}\beta,\iota_{4k}]=[\eta_{4k},\iota_{4k}]E^{4k-1}\beta=E(\tau_{4k-1}E^{4k-2}\beta)$. Assume that $E(\tau_{4k-1}E^{4k-2}\beta)=E(\eta_{4k-1}E(\beta'\alpha))$. Then, since the kernel of $E:\pi_{q+4k-2}(S^{4k-1})\to\pi_{q+4k-1}(S^{4k})$ coincides with $[\pi_q(S^{4k-1}),\iota_{4k-1}]$, we have

$$\tau_{4k-1}E^{4k-2}\beta=\eta_{4k-1}E(\beta'\alpha)+[\iota_{4k-1},\iota_{4k-1}]E^{4k-2}\gamma,\ \gamma\in\pi_q(S^{4k-1}).$$

By taking the Hopf invariant of both sides, we have a contradiction $\eta_{8k-3}^2(E^{4k-2}\beta)=0$.

Proposition 3.2. *Suppose that* $\theta\in\pi_q(S^{4k+1})$ *satisfies* $\eta_{8k-1}E^{4k-1}\theta\notin 2\pi_{q+4k-1}(S^{8k-1})$ *(e.g.* $\eta_{8k-1}E^{4k-1}\theta\neq 0$ *and the order of each element of* $\pi_{q+4k-1}(S^{8k-1})$ *equals 2 or is prime to that of* θ), *where* $q\leqq 8k-2$. *If* E^2: $\pi_{q+4k-2}(S^{q-2})\to\pi_{q+4k}(S^q)$ *is epic (e.g.* $q\geqq 4k+3$), *then* $[\theta,\iota_{4k+1}]\notin\theta\pi_{q+4k}(S^q)$.

Proof. Assume that $[\theta,\iota_{4k+1}]=\theta E^2\alpha$, $\alpha\in\pi_{q+4k-2}(S^{q-2})$. Since $[\theta,\iota_{4k+1}]=[\iota_{4k+1},\iota_{4k+1}]E^{4k}\theta=E(\tau_{4k}^{(1)}E^{4k-1}\theta)$ by Lemma 2.2 and since the kernel of $E:\pi_{q+4k-1}(S^{4k})\to\pi_{q+4k}(S^{4k+1})$ is generated by $[\pi_q(S^{4k}),\iota_{4k}]$, we have, for $\bar\theta$ with $E^2\bar\theta=\theta$,

$$\tau_{4k}^{(1)}E^{4k-1}\theta=E(\bar\theta\alpha)+[\iota_{4k},\iota_{4k}]E\gamma,\qquad \gamma\in\pi_{q+4k-2}(S^{8k-2}).$$

Taking the Hopf invariant of both sides yields $\eta_{8k-1}E^{4k-1}\theta\in 2\pi_{q+4k-1}(S^{8k-1})$, which contradicts our assumption.

Proposition 3.3. *Suppose* $E\theta\in\pi_q(S^n)$ *satisfies* $2E^{n-1}\theta\neq 0$, *where* n *is even and* $q\geqq n+1$. *Then* $[E\theta,\iota_n]\notin(E\theta)_*\pi_{n+q-1}(S^q)$.

Proof. Assume $[E\theta,\iota_n]=(E\theta)\alpha$; then $\alpha=E\alpha'$ for some $\alpha'\in\pi_{n+q-2}(S^{q-1})$. Taking the Hopf invariant, we get $2E^{n-1}\theta=0$.

Proposition 3.4. *Let* $n\equiv 2 \bmod 4$, $n\geqq 6$. *Then* $[\nu_n^2,\iota_n]\notin\nu_n^2\pi_{2n+5}(S^{n+6})$.

Proof. By Proposition 11.11, ii) of Toda [13], there is a $\tilde\nu_n\in\pi_{2n+3}(S^{n-2})$ such that $[\nu_n^2,\iota_n]=E^2\tilde\nu_n$ and $H(\tilde\nu_n)\equiv\varepsilon_{2n-5}$. Assume that $[\nu_n^2,\iota_n]=\nu_n^2E^2\alpha$, which implies that there is an integer x such that $E\tilde\nu_n=E(\nu_{n-2}^2E\alpha)+x[\sigma_{n-1},\iota_{n-1}]=E(\nu_{n-2}^2E\alpha)+x(E\tau_{n-2}^{(1)})\sigma_{2n-3}$ by Lemma 2.2. It follows that

$$\tilde\nu_n=\nu_{n-2}^2E\alpha+x\tau_{n-2}^{(1)}\sigma_{2n-4}+y[\bar\nu_{n-2},\iota_{n-2}]+z[\varepsilon_{n-2},\iota_{n-2}]$$

for some integers y and z. This leads to a contradiction $\varepsilon_{2n-5}\equiv x\eta_{2n-5}\sigma_{2n-4}=x(\bar\nu_{2n-5}+\varepsilon_{2n-5})\,\eta_{2n-5}\sigma_{2n-4}$ for $n\geqq 10$.

We now proceed to prove the theorem. Take $\beta=\iota_{4k+1}$ $(k\geqq 3)$ in Proposition 3.1. Since any element of $\pi_{8k-1}(S^{4k})$ is expressible as $E\gamma+x[\iota_{4k},\iota_{4k}]$, we see that $E^2:\pi_{8k-2}(S^{4k-1})\to\pi_{8k}(S^{4k+1})$ is epic. Thus the assertion for $[\eta_{4k},\iota_{4k}]$ $(k>2)$ follows. The case $k=2$ follows from the fact that $[\eta_8,\iota_8]=(E\sigma')\eta_{15}$ and $\eta_8\sigma_9=(E\sigma')\eta_{15}+\bar\nu_8+\varepsilon_8$. Applying Proposition 3.1 to $\beta=\eta_{4k+1}$ $(k\geqq 2)$, σ_{4k+1} $(k\geqq 3)$, μ_{4k+1} $(k\geqq 3)$, ρ_{4k+1} $(k\geqq 5)$, η_{4k+1}^* $(k\geqq 6)$,

$\bar{\mu}_{4k+1}$ $(k\geqq 6)$ and observing relations $\eta^3_{8k-3}=4\nu_{8k-3}$, $\eta^2_{8k-3}\sigma_{8k-1}=\nu^3_{8k-3}+\eta_{8k-3}\varepsilon_{8k-2}$, $\eta^2_{8k-3}\mu_{8k-1}=4\zeta_{8k-3}$, $\eta^2_{8k-3}\rho_{8k-1}=\sigma_{8k-3}\eta_{8k+4}\mu_{8k+5}$, $\eta^2_{8k-3}\eta^*_{8k-1}=4\nu^*_{8k-3}$, $\eta^2_{8k-3}\bar{\mu}_{8k-1}=4\bar{\zeta}_{8k-3}$, the cases involving η_{4k} are settled.

We may apply Proposition 3.2 by taking for θ η_{4k+1} $(k\geqq 1)$, σ_{4k+1} $(k\geqq 3)$, $\eta_{4k+1}\sigma_{4k+2}$ $(k\geqq 3)$, $\bar{\nu}_{4k+1}$ $(k\geqq 3)$, ε_{4k+1} $(k\geqq 3)$, μ_{4k+1} $(k\geqq 3)$, κ_{4k+1} $(k\geqq 5)$, $\bar{\mu}_{4k+1}$ $(k\geqq 5)$, ρ_{4k+1} $(k\geqq 5)$, $\eta_{4k+1}\rho_{4k+2}$ $(k\geqq 5)$, ω_{4k+1} $(k\geqq 5)$, $\bar{\kappa}_{4k+1}$ $(k\geqq 6)$. Here we note that $\eta_{8k-1}\kappa_{8k}=\bar{\varepsilon}_{8k-1}\notin 2\pi_{8k+14}(S^{8k-1})=\{2\rho_{8k-1}\}+Z_{15}$, $\eta_{8k-1}\rho_{8k}=\sigma_{8k-1}\mu_{8k+6}$, $\eta_{8k-1}\bar{\mu}_{8k}\notin 2\pi_{8k+17}(S^{8k-1})=\{2\nu^*_{8k-1}\}$.

We shall show that $[\nu_{8k+1},\iota_{8k+1}]\notin\nu_{8k+1}\pi_{16k+4}(S^{8k+4})$, $k\geqq 1$. Assume $[\nu_{8k+1},\iota_{8k+1}]=\nu_{8k+1}\alpha$. Since $[\iota_{8k},\iota_{8k}]$ is of infinite order, we may write $\alpha=E^4\alpha'$, $\alpha'\in\pi_{16k}(S^{8k})$. By Lemma 2.4, $E^3\bar{\tau}_{8k-2}=E^3(\nu_{8k-2}E\alpha')$, and $\pi_{16k+5}(S^{16k+1})=\pi_{16k+4}(S^{16k-1})=0$ gives

$$\pi_{16k+1}(S^{8k-2})\xrightarrow{E}\!\!\!\!\twoheadrightarrow\pi_{16k+2}(S^{8k-1})\rightarrowtail\!\!\!\!\xrightarrow{E}\!\!\!\!\twoheadrightarrow\pi_{16k+3}(S^{8k})\rightarrowtail\!\!\!\!\xrightarrow{E}\pi_{16k+4}(S^{8k+1}),$$

in which the kernel of the first E is generated by $[\nu^2_{8k-2},\iota_{8k-2}]$, so that we get $\bar{\tau}_{8k-2}=\nu_{8k-2}E\alpha'+x[\nu^2_{8k-2},\iota_{8k-2}]$. This is a contradiction, because the Hopf invariant of the right hand side is 0.

Next we show that $[\nu_{8k+3},\iota_{8k+3}]\notin\nu_{8k+3}\pi_{16k+8}(S^{8k+6})$, $k\geqq 1$. Assume that $[\nu_{8k+3},\iota_{8k+3}]=\nu_{8k+3}\alpha$. Since $\pi_{16k+5}(S^{8k+3})$ is finite, $E:\pi_{16k+4}(S^{8k+2})\to\pi_{16k+5}(S^{8k+3})$ is epic, hence $\alpha=E^4\alpha'$. By Lemma 2.3, we have $E^3(\tau^{(3)}_{8k}\nu_{16k+2})=E^3(\nu_{8k}E\alpha')$. Since $\pi_{16k+9}(S^{16k+5})=\pi_{16k+8}(S^{8k+3})=0$, it follows that $\tau^{(2)}_{8k}\nu_{16k+2}=\nu_{8k}E\alpha'+x[\nu^2_{8k},\iota_{8k}]$. By taking the Hopf invariant, a contradiction arises.

Finally we show that, for $n\equiv 2 \bmod 4$, $n\geqq 10$, $[8\sigma_n,\iota_n]\notin 8\sigma_n\pi_{2n+6}(S^{n+7})$, which completes the proof of the theorem. By Proposition 11.11, (ii) of Toda [13] there exists $\beta\in\pi_{2n+4}(S^{n-2})$ such that $[8\sigma_n,\iota_n]=E^2\beta$, $H(\beta)\in\{\eta_{2n-5},2\iota_{2n-4},8\sigma_{2n-4}\}$, i.e., $H(\beta)\equiv\mu_{2n-5} \bmod \eta_{2n-5}\pi_{2n+4}(S^{2n-4})$. Assume that $[8\sigma_n,\iota_n]=E^2(8\sigma_{n-2}E\alpha)$, $\alpha\in\pi_{2n+3}(S^{n+4})$. Then

$$E\beta\equiv E(8\sigma_{n-2}E\alpha) \bmod \{[\nu_{n-1},\iota_{n-1}],[\varepsilon_{n-1},\iota_{n-1}]\}.$$

Since the indeterminacy is equal to $\{(E\tau^{(1)}_{n-2})\bar{\nu}_{2n-3},(E\tau^{(1)}_{n-2})\varepsilon_{2n-3}\}$ by Lemma 2.2, we get

$$\beta\equiv 8\sigma_{n-2}E\alpha \bmod \{\tau^{(1)}_{n-2}E\pi_{2n+3}(S^{2n-5}),[\iota_{n-2},\iota_{n-2}]\pi_{2n+3}(S^{2n-5})\},$$

which leads to a contradiction in taking the Hopf invariant (Note that, for $n=10$ $\pi_{24}(S^{15})=(Z_2)^3$).

Added in proof. Using Proposition 11.10, i) of [13] we can show that, for $E^2\alpha\in\pi_q(S^n)$ of order 2 with $\eta_{2n-3}(E^{n-1}\alpha)\neq 0$, where $n\equiv 0 \bmod 4$, $n+2\leqq q\leqq 2n-5$, we have $[E^2\alpha,\iota_n]\notin(E^2\alpha)\pi_{n+q-1}(S^q)$. This may be applied to ε_n $(n\geqq 16)$, $\bar{\nu}_n(n\geqq 16)$, μ_n $(n\geqq 16)$, κ_n $(n\geqq 20)$ and $\bar{\mu}_n$ $(n\geqq 24)$.

References

[1] P. J. Hilton: A note on the P-homomorphism in homotopy groups of spheres. Proc. Cambridge Phil. Soc., **51**, 230–233 (1955).

[2] W. C. Hsiang, J. Levine, and R. H. Szczarba: On the normal bundle of a homotopy sphere embedded in Euclidean space. Topology, **3**, 173–181 (1965).
[3] I. M. James: On the iterated suspension. Quart. J. Math., **5**, 1–10 (1954).
[4] ——: Note on cup-products. Proc. Amer. Math. Soc., **8**, 374–383 (1957).
[5] I. M. James and J. H. C. Whitehead: The homotopy theory of sphere bundles over spheres. I. Proc. London Math. Soc., **4**, 196–218 (1954).
[6] M. A. Kervaire: Some nonstable homotopy groups of Lie groups. Illinois J. Math., **4**, 161–169 (1960).
[7] L. Kristensen and I. Madsen: Note on Whitehead products in spheres. Math. Scand., **21**, 301–314 (1967).
[8] M. Mahowald: Some Whitehead products in (S^n). Topology, **4**, 17–26 (1965).
[9] M. Mimura: On the generalized Hopf homomorphism and the higher composition, Part II. $\pi_{n+i}(S^n)$ for $i=21$ and 22. J. Math. Kyoto Univ., **4**, 301–326 (1965).
[10] M. Mimura and H. Toda: The $(n+20)$-th homotopy groups of n-spheres. J. Math. Kyoto Univ., **3**, 37–58 (1963).
[11] K. Ôguchi: A generalization of secondary composition and its application. J. Fac. Sci., Univ. of Tokyo, Sec. I., **10**, 29–79 (1963).
[12] G. F. Paechter: The groups $\pi_r(V_{n,m})$. I. Quart. J. Math., **7**, 249–268 (1956).
[13] H. Toda: Composition Methods in Homotopy Groups of Spheres. Princeton (1962).