8. *Paracompactness of Topological Completions*

By Tadashi ISHII
Shizuoka University

(Comm. by Kenjiro SHODA, M. J. A., Jan. 12, 1974)

1. Introduction. All spaces are assumed to be completely regular T_2 unless otherwise specified. This paper is mainly concerned with paracompactness of the completion $\mu(X)$ of a space X with respect to its finest uniformity μ. Such completion of a space X is called the topological completion of X (or the completion in the sense of Dieudonné). Following Morita [12], a space X is said to be pseudo-paracompact (resp. pseudo-Lindelöf etc.) if $\mu(X)$ is paracompact (resp. Lindelöf etc.). Since for any M-space X $\mu(X)$ is a paracompact M-space ([12]), every M-space is pseudo-paracompact.

The purpose of this paper is to study some properties of pseudo-paracompact spaces. The details will be published elsewhere.

2. Characterizations of pseudo-paracompact spaces. An open covering $\mathfrak{O}=\{O_\alpha\}$ of a space X is said to be extendable to $\mu(X)$ if there exists an open covering $\tilde{\mathfrak{O}}=\{\tilde{O}_\alpha\}$ of $\mu(X)$ such that $O_\alpha=\tilde{O}_\alpha\cap X$ for each α. We note that every normal open covering of X is extendable to $\mu(X)$ as a normal open covering (cf. [9, (I) Lemma 8 and (II) Lemma 1]).

Now let $\{\mathfrak{U}_\lambda \mid \lambda\in\Lambda\}$ be the set of all the normal open coverings of a space X. A filter $\mathfrak{F}=\{F_\alpha\}$ in X is said to be weakly Cauchy with respect to the uniformity μ if for any $\lambda\in\Lambda$ there exists $U\in\mathfrak{U}_\lambda$ such that $U\cap F_\alpha\neq\phi$ for every α. In other words, a filter $\mathfrak{F}$ is weakly Cauchy if for any $\lambda\in\Lambda$ there exists a stronger filter $\mathfrak{F}_\lambda$ than $\mathfrak{F}$ such that $L\subset U$ for some $U\in\mathfrak{U}_\lambda$ and $L\in\mathfrak{F}_\lambda$. We state first the necessary and sufficient conditions for a space X to be pseudo-paracompact, some of which are the modifications of Corson's theorem [1] for the characterizations of paracompact spaces.

Theorem 2.1. *For a space X, the following conditions are equivalent.*

(a) *X is pseudo-paracompact.*

(b) *Every open covering of X which is extendable to $\mu(X)$ is a normal covering.*

(c) *The product of X with every compact space is pseudo-normal.*

(d) *Every weakly Cauchy filter in X with respect to μ is contained in some Cauchy filter with respect to μ.*

(e) *If $\mathfrak{F}$ is a filter in X such that the image of $\mathfrak{F}$ has a cluster* *point in any metric space into which X is continuously mapped, then $\mathfrak{F}$ is contained in some Cauchy filter with respect to μ.*

The equivalence of (a) and (d) was essentially proved by Howes [4].

The following example shows that there exists a space which is strongly normal (that is, countably paracompact and collectionwise normal) but not pseudo-paracompact.

Example 2.2. Let X be a subspace of the product $\prod_{\alpha\in A} R_\alpha$ which consists of those points which have at most a countable number of non-zero coordinates, where A is uncountable index set and R_α is the real line for each $\alpha\in A$. In [2], Corson proved that (a) X is strongly normal and that (b) $\upsilon(X)=\prod_{\alpha\in A} R_\alpha$, where $\upsilon(X)$ denotes the realcompactification of a space X. But we can prove that $\mu(X)=\upsilon(X)$. Hence X is not even pseudo-normal, since $\prod_{\alpha\in A} R_\alpha$ is not normal ([14]).

3. Some results related to pseudo-paracompactness. We shall state first the sum theorems of pseudo-paracompact spaces, with which the following two theorems are concerned.

Theorem 3.1. *If there exists a normal open covering $\mathfrak{U}=\{U_\alpha\}$ of X such that each subspace U_α is pseudo-paracompact, then X is pseudo-paracompact.*

Theorem 3.2. *Let $\{F_\alpha \mid \alpha\in\Omega\}$ be a locally finite closed covering of X such that each subspace F_α is pseudo-paracompact. If X is strongly normal, then X is pseudo-paracompact.*

We don't know whether Theorem 3.2 is valid or not in case X is not strongly normal.

Now let $f: X\to Y$ be a continuous map. Then there exists its extension $\beta(f): \beta(X)\to\beta(Y)$, where $\beta(S)$ denotes the Stone-Čech compactification of a space S, and it is known that $\beta(f)$ carries $\mu(X)$ into $\mu(Y)$ ([12]). We denote this map by $\mu(f)$. A continuous map f from a space X onto a space Y is called a WZ-map (Ishiwata [6]), a Z-map and a quasi-perfect (resp. perfect) map if it satisfies (1), (2) and (3) below respectively:

(1) $\beta(f)^{-1}(y)=\mathrm{cl}_{\beta(Y)} f^{-1}(y)$ for any $y\in Y$.

(2) $f(Z)$ is closed in Y for each zero-set Z of X.

(3) f is a closed map such that $f^{-1}(y)$ is countably compact (resp. compact) for each $y\in Y$.

Every closed map is a Z-map, and every Z-map is a WZ-map ([6]).

The following theorem is concerned with a relation between f and $\mu(f)$, and it is useful to show that the preimages of paracompact spaces under quasi-perfect maps are pseudo-paracompact.

Theorem 3.3. *If f is a quasi-perfect map from a space X onto a topologically complete space Y, then $\mu(f): \mu(X)\to Y$ is perfect. More generally, if f is a WZ-map from a space X onto a topologically complete space Y such that $f^{-1}(y)$ is relatively pseudo-compact (that is,* *every real-valued continuous function on X is bounded on $f^{-1}(y)$) for each $y \in Y$, then $\mu(f): \mu(X) \to Y$ is perfect.*

Corollary 3.4. *If f is a quasi-perfect map from a space X onto a paracompact space Y, then X is pseudo-paracompact. More generally, if f is a WZ-map from a space X onto a paracompact space Y such that $f^{-1}(y)$ is relatively pseudo-compact for each $y \in Y$, then X is pseudo-paracompact.*

In case the fibers $\{f^{-1}(y)\}$ are not necessarily relatively pseudo-compact, we have the following theorem.

Theorem 3.5. *If there exists a Z-map f from a space X onto a paracompact q-space Y (in the sense of Michael [8]) such that $f^{-1}(y)$ is pseudo-paracompact for each $y \in Y$, then X is pseudo-paracompact.*

We note here that if f is a Z-map from a space X onto a q-space Y, then $\mathfrak{B}f^{-1}(y)$ (=the boundary of $f^{-1}(y)$) is relatively pseudo-compact for each $y \in Y$. This is a slight modification of [8, Theorem 2.1]. Hence, to prove Theorem 3.5, it is sufficient to prove the following theorem.

Theorem 3.6. *If there exists a Z-map f from a space X onto a paracompact space Y such that $\mathfrak{B}f^{-1}(y)$ is relatively pseudo-compact and $f^{-1}(y)$ is pseudo-paracompact for each $y \in Y$, then X is pseudo-paracompact.*

This theorem can be deduced from the following lemma.

Lemma 3.7. *If there exists a Z-map f from a space X onto a paracompact space Y such that $\mathfrak{B}f^{-1}(y)$ is relatively pseudo-compact and that, for any open covering $\mathfrak{O}$ of X which is extendable to $\mu(X)$, $f^{-1}(y) \cap \mathfrak{O}$ is a normal covering of the subspace $f^{-1}(y)$ for each $y \in Y$, then X is pseudo-paracompact.*

As a direct consequence of Theorem 3.5, we have the following corollary.

Corollary 3.8. *Let f be a closed (or Z-) map from a space X onto a metric space Y. Then X is pseudo-paracompact in the following cases.*

(a) *$f^{-1}(y)$ is an M-space for each $y \in Y$.*

(b) *$f^{-1}(y)$ is paracompact for each $y \in Y$.*

In Theorem 3.5, we can not exclude the assumption that Y is a q-space. This is shown by making use of the quotient map f from the space Π onto the quotient space Π/D (cf. [3, 6Q]), since f is a closed map and $f^{-1}(y)$ is a metric space for each $y \in \Pi/D$. Moreover in Theorem 3.5 we can not replace 'Z-map' by 'open map'. To see this, let X be a metric space and Y a paracompact space such that the product $X \times Y$ is not normal ([7]), and let $\varphi: X \times Y \to X$ be the projection map. Then φ is an open map from $X \times Y$ onto a metric space X

such that $\varphi^{-1}(x)$ is paracompact for each $x \in X$. But $X \times Y$ is not even pseudo-normal, since $X \times Y$ is topologically complete.

Concerning Corollary 3.8, we note that if X is the preimage of a metric space Y under a closed map f such that $f^{-1}(y)$ is an M-space (resp. paracompact), then X is not necessarily an M-space (resp. paracompact). Hoshina proved the validity of the paracompact case by making use of the quotient map $\varphi: \psi \to \psi/D$ (cf. [3, 5I]). But this example shows that the case for M-spaces is also valid.

The problem whether the images (or preimages) of pseudo-paracompact spaces under perfect maps are also pseudo-paracompact or not is unsolved, but we can prove the following theorem.

Theorem 3.9. *Let $f: X \to Y$ be a quasi-perfect map. If X is strongly normal and pseudo-paracompact, then so is Y.*

We note that Theorem 3.2 is easily deduced from this theorem.

The following theorem is concerned with the necessary and sufficient conditions for a space X in order that $\mu(X)$ be locally compact and paracompact.

Theorem 3.10. *For a space X, the following conditions are equivalent.*

(a) *X is pseudo-locally-compact and pseudo-paracompact.*

(b) *There exists a normal open covering $\mathfrak{U} = \{U_\alpha\}$ such that each U_α is relatively pseudo-compact in X.*

(c) *There exists a normal sequence $\{\mathfrak{U}_n\}$ of open coverings of X such that for each $x \in X$ $\mathrm{St}(x, \mathfrak{U}_{k(x)})$ is relatively pseudo-compact in X for some $k(x)$.*

(d) *There exists a Z-map f from X onto a locally compact metric space T such that $f^{-1}(y)$ is relatively pseudo-compact for each $y \in T$.*

(e) *There exists a Z-map f from X onto a locally compact and paracompact space Y such that $f^{-1}(y)$ is relatively pseudo-compact for each $y \in Y$.*

(f) *There exists a WZ-map f from X onto a locally compact and paracompact space such that $f^{-1}(y)$ is relatively pseudo-compact for each $y \in Y$.*

The equivalence of (a) and (b) is due to Morita [13], who proved also the equivalence of (a) and (d) independently.

4. Pseudo-Lindelöf property. For a space X we denote by υ the uniformity of X which consists of all the countable normal open coverings of X. As for the characterizations of pseudo-Lindelöf spaces, we have the following theorem.

Theorem 4.1. *For a space X, the following conditions are equivalent.*

(a) *X is pseudo-Lindelöf.*

(b) *X is pseudo-paracompact and any normal open covering of X has a countable subcovering.*

(c) *Every open covering of X which is extendable to $\mu(X)$ has a countable subcovering.*

(d) *Every weakly Cauchy filter in X with respect to υ is contained in some Cauchy filter with respect to μ.*

(e) *If $\mathfrak{F}$ is a filter in X such that the image of $\mathfrak{F}$ has a cluster point in any separable metric space into which X is continuously mapped, then $\mathfrak{F}$ is contained in some Cauchy filter with respect to μ.*

The equivalence of (a) and (b) was proved by Howes [4].

As is easily seen from (c) in Theorem 4.1, the image of a pseudo-Lindelöf space under a continuous map is pseudo-Lindelöf. This result was first pointed out by K. Morita. Therefore it follows that if a space X is the countable union of the pseudo-Lindelöf subspaces then X is also pseudo-Lindelöf.

Corresponding to Theorem 3.5, we can prove the following theorem.

Theorem 4.2. *If there exists a Z-map f from a space X onto a Lindelöf space Y such that $f^{-1}(y)$ is pseudo-Lindelöf for each $y \in Y$, then X is pseudo-Lindelöf.*

To prove this theorem, we make use of the following lemma.

Lemma 4.3. *If there exists a Z-map f from a space X onto a Lindelöf space Y such that, for any open covering $\mathfrak{O}$ of X which is extendable to $\mu(X)$, $f^{-1}(y) \cap \mathfrak{O}$ has a countable subcovering for each $y \in Y$, then X is pseudo-Lindelöf.*

References

[1] H. H. Corson: The determination of paracompactness by uniformities. Amer. J. Math., **80**, 185–190 (1958).

[2] ——: Normality of subsets of product spaces. Amer. J. Math., **81**, 785–796 (1959).

[3] L. Gillman and M. Jerison: Rings of Continuous Functions. Van Nostrand, Princeton, N. J. (1960).

[4] N. R. Howes: On completeness. Pacific J. Math., **38**, 431–440 (1971).

[5] M. Katětov: Extensions of locally finite coverings. Colloq. Math., **6**, 141–151 (1958).

[6] T. Isiwata: Mappings and spaces. Pacific J. Math., **20**, 455–480 (1967).

[7] E. Michael: The product of a normal space and a metric space need not be normal. Bull. Amer. Math. Soc., **69**, 375–376 (1963).

[8] ——: A note on closed maps and compact sets. Israel J. Math., **2**, 173–176 (1964).

[9] K. Morita: Simple extension of a space with respect to a uniformity. I, II. Proc. Japan Acad., **27**, 64–72, 130–137 (1951).

[10] ——: Paracompactness and product spaces. Fund. Math., **50**, 223–236 (1962).

[11] K. Morita: Products of normal spaces with metric spaces. Math. Ann., **154**, 365–382 (1964).
[12] ——: Topological completions and M-spaces. Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, **10**, 271–288 (1970).
[13] ——: On the dimension of the product of Tychonoff spaces. Gen. Top. and its appl., **3**, 125–133 (1973).
[14] A. H. Stone: Paracompactness and product spaces. Bull. Amer. Math. Soc., **54**, 977–982 (1948).
[15] H. Tamano: On paracompactness. Pacific J. Math., **10**, 1043–1047 (1960).