32. *On Certain L^2-well Posed Mixed Problems for Hyperbolic System of First Order*

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1. Introduction and Theorem. Let P be a x_0-strictly hyperbolic $2p\times 2p$-system of differential operators of first order defined over a C^∞-cylinder $R^1\times\Omega\subset R^{n+1}$. Let B be a $p\times 2p$-system of functions defined on the boundary Γ of $R^1\times\Omega$. We consider the following mixed problems under certain conditions:

$$
\begin{aligned}
&P(x,D)u=f \quad && x\in R^1\times\Omega \quad && (x_0>0),\\
&B(x)u=g && x\in\Gamma && (x_0>0),\\
&u=h && \text{on } x_0=0 &&
\end{aligned}
$$

where $\sqrt{-1}D=\left(\frac{\partial}{\partial x_0},\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}\right)$.

For the sake of simplicity of descriptions, we may only consider the case where $\Omega=\{x_n>0\}$, by the localization process. Then our assumptions are the following:

(I) α) The coefficients of P and B are real, belong to $C^\infty(R^1\times\bar\Omega)$ and constant outside some compact set of $R^1\times\bar\Omega$.

β) For P, it satisfies the $\sharp$ condition with respect to Γ and for fixed real (x,τ,σ) there is at most one real double root λ of $|P|(x,\tau,\sigma,\lambda)=0$ where $x\in\Gamma$. Furthermore it is non-characteristic with respect to Γ and it is normal, i.e.

$$|P|(x,0,\sigma,\lambda)\neq 0$$

for any real $(\sigma,\lambda)\neq 0$.

γ) The p row-vectors of $B(x)$ are linearly independent, where $x\in\Gamma$.

(II) α) If the Lopatinsky determinant $R(x_0,\tau_0,\sigma_0)=0$ for a real point (x_0,τ_0,σ_0) such that there are no real double roots λ of $|P|(x_0,\tau_0,\sigma_0,\lambda)=0$, then

$$|R(x_0,\tau_0-i\gamma,\sigma_0)|\geq 0(\gamma^1)\qquad(\gamma>0).$$

Furthermore if there is at least one real simple root $\lambda(x_0,\tau_0,\sigma_0)$, the zero set of $R(x,\tau\pm i\gamma,\sigma)$ in some neighborhood $U(x_0,\tau_0,\sigma_0)$ is in the set $\{\gamma=0\}$.

β) If $R(x_0,\tau_0,\sigma_0)=0$ for a real point (x_0,τ_0,σ_0) such that there are real double roots λ of $|P|(x_0,\tau_0,\sigma_0,\lambda)=0$, then

$$|R(x_0,\tau_0-i\gamma,\sigma_0)|\geq 0(\gamma^{1/2})\qquad(\gamma>0).$$

Furthermore if there is at least one real simple root λ, the rank of the

Hessian of $R(x,\tau,\sigma)$ at its zeros in some $U(x_0,\tau_0,\sigma_0)$ is equal to

$$\text{codim. of } \{R(x,\tau,\sigma)=0\} \qquad \text{in } R^{2n-1}.$$

Where the zero set of $R(x,\tau,\sigma)$ in some $U(x_0,\tau_0,\sigma_0)$ is preassumed to be a regular submanifold of R^{2n}.

γ) Moreover, if there is at least one non-real root λ of $|P|(x_0,\tau_0,\sigma_0,\lambda)=0$ for the point (x_0,τ_0,σ_0) which satisfies the condition β), then for some smooth and non-singular matrix $S(x,\tau-i\gamma,\sigma)$ with $\gamma\geq 0$ defined on some $U(x_0,\tau_0,\sigma_0)$, one of the corresponding coupling coefficients $b_{\text{II}\,\text{II}}(x,\tau,\sigma)$ is real whenever τ and $\lambda^{+}_{\text{II}}(x,\tau,\sigma)$ are real (For definitions, see § 2).

(III) Any constant coefficients problems frozen the coefficient at boundary are L^2-well posed.

Then we have the following

Theorem. *Under assumptions* (I), (II), (III), *the mixed problem is L^2-well posed.*

The aim of the present note is to describe the outline of our proof of the above assertion. Here we use essentially the conception of reflection coefficients ([1], [2]) and modifying Kreiss' consideration ([4]) we make use of the micro-localization of the characterization for L^2-well posed mixed problem of order two ([1], [3] and [7]).

2. The outline of the proof. Considering the assumption (I) let $S(x,\tau-i\gamma,\sigma)$ $(\gamma\geq 0)$ be a smooth, non-singular matrix defined on some neighborhood $U(x_0,\tau_0,\sigma_0)$ such that

$$S^{-1}PS=ED_n-A(x,\tau-i\gamma,\sigma)$$

where

$$A=\begin{pmatrix} \lambda^{+}_{\text{I}} & & & & \\ & \lambda^{-}_{\text{I}} & & & \\ & & A_{\text{II}} & & \\ & & & A^{+}_{\text{III}} & \\ & & & & A^{-}_{\text{III}} \end{pmatrix},$$

$$\lambda^{\pm}_{\text{I}}=\begin{pmatrix} \cdot & & & & \\ & \cdot & & & \\ & & \lambda^{\pm}_{i} & & \\ & & & \cdot & \\ & & & & \cdot \end{pmatrix}, \quad i\in \text{I}, \quad |\text{I}|=r,$$

$\lambda^{\pm}_{i}$ are real for $\gamma=0$, and $\operatorname{Im}\lambda^{+}_{i}$ $(\operatorname{Im}\lambda^{-}_{i})>0$ (<0) respectively if $\gamma>0$. Next for $\tau_0=\tau_0(x,\sigma)$

$$A_{\text{II}}(x,\tau_0,\sigma)=\begin{pmatrix} a(x,0,\sigma) & 1 \\ 0 & a(x,0,\sigma)\end{pmatrix}.$$

Here we may restrict ourself to the case where the eigenvalue of $A_{\text{II}}(x,\tau,\sigma)$ are described by the following form in some $U(x_0,\tau_0,\sigma_0)$;

$$\lambda^{\pm}_{\text{II}}=a(x,\zeta,\sigma)\mp\sqrt{\zeta}b(x,\zeta,\sigma) \qquad (\sqrt{1}=1),$$

$a(x,\zeta,\sigma)$, $b(x,\zeta,\sigma)$ are real when ζ is real, $b(x,\zeta,\sigma)\neq 0$, $\tau_0=\tau_0(x_0,\sigma_0)$,

$\tau=\zeta+\tau_0(x,\sigma)$ and $\tau_0(x,\sigma)$ is real and positive.
Furthermore $A^{\pm}_{\mathrm{III}}$ have only non-real eigenvalues for any $\gamma\geq 0$ and the ones of A^{+}_{III} have positive imaginary parts.

Let $BS=(V^{+}_{\mathrm{I}}, V^{-}_{\mathrm{I}}, V'_{\mathrm{II}}, V''_{\mathrm{II}}, V^{+}_{\mathrm{III}}, V^{-}_{\mathrm{III}})$. Where $V^{\pm}_{\mathrm{I}}$ are $(p\times r)$-matrices, V'_{II}, V''_{II} are p-vectors and $V^{\pm}_{\mathrm{III}}$ are $(p\times s)$-matrices respectively $(2r+2+2s=2p)$.

Let $S_{\mathrm{II}}=\begin{pmatrix} 1 & 0 \\ \dfrac{\lambda^{+}_{\mathrm{II}}-h_{11}\zeta-a}{1+h_{12}\zeta}, & 1\end{pmatrix}$, $\qquad a=a(x,0,\sigma)$

and let

$$S'=\begin{pmatrix} E_{2r} & & \\ & S_{\mathrm{II}} & \\ & & E_{2s}\end{pmatrix},$$

where h_{ij} are the functions derived from $A_{\mathrm{II}}(x,\tau-i\gamma,\sigma)$. Furthermore we denote $B\cdot S\cdot S'$ by

$$(V^{+}_{\mathrm{I}}, V^{-}_{\mathrm{I}}, V^{+}_{\mathrm{II}}, V^{-}_{\mathrm{II}}, V^{+}_{\mathrm{III}}, V^{-}_{\mathrm{III}})(x,\tau,\sigma).$$

Then from our assumptions we obtain the following Lemmas. In particular from (I) γ), (II) α) and (III), we see the following

Lemma 2.1. *If for real (x_0,τ_0,σ_0) there exist no real double roots λ, then there is neighborhood $U(x_0,\tau_0,\sigma_0)$ where*

i) *For some $V^{-}_{3,i}$ the determinant*

$$|V^{+}_{\mathrm{I}}, V^{+}_{3,1}, \cdots, V^{+}_{3,i-1}, V^{-}_{3,i}, V^{+}_{3,i+1}, \cdots, V^{+}_{3,s}|\neq 0$$

where $V^{+}_{\mathrm{III}}=(V^{+}_{3,1},\cdots,V_{3,s})$, $s=p-\gamma$, $V^{+}_{3,i}$ are p-column vectors (Here after let $i=1$.).

ii) *For some $V^{+}_{3,1}$ it belongs to the linear subspace $L(V^{+}_{3,2},\cdots,V^{+}_{3,s})$ spanned by the vectors $V^{+}_{3,2},\cdots,V_{3,s}$.*

iii) *The column vectors of V^{-}_{I} belong to $L(V^{+}_{\mathrm{I}}, V^{+}_{3,2},\cdots,V^{+}_{3,s})$. But* ii) *and* iii) *are only valid at the points $\in U(x_0,\tau_0,\sigma_0)$ such that the Lopatinsky* det. $|V^{+}_{\mathrm{I}}, V^{+}_{\mathrm{III}}|(x,\tau,\sigma)=c(\tau-\tau(x,\sigma))=0$ $(c\neq 0)$ *and where $\tau(x,\sigma)$ is real whenever V^{+}_{I} present.*

From (II) β) and γ) we see the following

Lemma 2.2. *Let (x_0,τ_0,σ_0) be a real point such that there exists a real double root λ. Let $|V^{+}_{\mathrm{I}}, V^{+}_{\mathrm{II}}, V^{+}_{\mathrm{III}}|(x_0,\zeta,\sigma_0)=0$, where we consider ζ as a new variable instead of τ. Then*

i) $\zeta=0$.

ii) *Let $\zeta^{1/2}=\eta$, then*

$$|V^{+}_{\mathrm{I}}, V^{+}_{\mathrm{II}}, V^{+}_{\mathrm{III}}|=C(\eta-\eta(x,\sigma)) \qquad (c\neq 0)$$

in some $U(x_0,\tau_0,\sigma_0)$, where $\eta(x,\sigma)$ may take complex values.

Under the assumption of Lemma 2.2 we see the following Lemmas.

Lemma 2.3. i) *The coupling coefficient*

$$b_{\mathrm{II\,II}}(x_0,-i\gamma,\sigma_0)=\frac{|V^{+}_{\mathrm{I}}, V^{-}_{\mathrm{II}}, V^{+}_{\mathrm{III}}|}{|V^{+}_{\mathrm{I}}, V^{+}_{\mathrm{II}}, V^{+}_{\mathrm{III}}|}(x_0,-i\gamma,\sigma_0)$$
$$=0(\gamma^{-1/2}) \qquad (\gamma>0).$$

ii) *Let* $Q(x,\zeta,\sigma)$ *be* $\dfrac{a_{11}+a_{21}b_{\mathrm{II\,II}}}{a_{12}+a_{22}b_{\mathrm{II\,II}}}$, *then it is* $\dfrac{|V_{\mathrm{I}}^{+}, V'_{\mathrm{II}}, V_{\mathrm{III}}^{+}|}{|V_{\mathrm{I}}^{+}, V''_{\mathrm{II}}, V_{\mathrm{III}}^{+}|}$, *where* $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = S_{\mathrm{II}}^{-1}$.

Now from Lemma 2.3 and (III) we obtain the following

Lemma 2.4.

i) $|V_{\mathrm{I}}^{+}, V''_{\mathrm{II}}, V_{\mathrm{III}}^{+}| \neq 0$.

ii) $V'_{\mathrm{II}} \in L(V_{\mathrm{III}}^{+})$ *on* $\zeta=\eta(x,\sigma)=0$.

iii) $V_{\mathrm{I}}^{-} \in L(V_{\mathrm{I}}^{+}, V_{\mathrm{III}}^{+})$ *on* $\zeta=\eta(x,\sigma)=0$.

iv) $V''_{\mathrm{II}} - QV''_{\mathrm{II}} \in L(V_{\mathrm{I}}^{+}, V_{\mathrm{III}}^{+})$.

From (II) β), γ), (III) and the definition of Q we see the following

Lemma 2.5. i) *The above defined* $Q(x,\zeta,\sigma)$ *takes only real values, when* ζ *is real.*

ii) $\zeta=0$, $Q(x,0,\sigma)=0$ *are equivalent to* $R(x,\zeta,\sigma)=0$ *for* $\mathrm{Im}\,\zeta \leq 0$.

iii) $-Q(x,0,\sigma) \geq 0$.

From Lemma 2.4 we obtain the following

Lemma 2.6. *For* (x,ζ,σ) *belonging to some* $U(x_0,\tau_0,\sigma_0)$,

$$g=(V_{\mathrm{I}}^{+}, V''_{\mathrm{II}}, V_{\mathrm{III}}^{+})\begin{pmatrix} U_{\mathrm{I}}^{+}+(\zeta K'_{\mathrm{I\,II}}+K''_{\mathrm{I\,II}})U'+K_{\mathrm{II}}U_{\mathrm{I}}^{-} \\ U''_{\mathrm{II}}+QU'_{\mathrm{II}}+(\zeta K'_{\mathrm{II\,I}}+K''_{\mathrm{II\,I}})U_{\mathrm{I}}^{-} \\ U_{\mathrm{III}}^{+}+K_{\mathrm{III\,I}}U_{\mathrm{I}}^{-}+K_{\mathrm{III\,II}}U'_{\mathrm{II}} \end{pmatrix} + V_{\mathrm{III}}^{-}U_{\mathrm{III}}^{-},$$

where $u=(U_{\mathrm{I}}^{+}, U_{\mathrm{I}}^{-}, U'_{\mathrm{II}}, U''_{\mathrm{II}}, U_{\mathrm{III}}^{+}, U_{\mathrm{III}}^{-})$. *Moreover the components of* $K''_{\mathrm{I\,II}}$ and $K''_{\mathrm{II\,I}}$ *are zero, whenever* $\zeta=0$ *and* $\eta(x,\sigma)=0$.

From Lemma 2.1 we obtain an a priori L^2-estimate in the case where there is no real double root λ. On the other hand if there is at least one real double root λ, we see from Lemma 2.5 and by some modifications of Kreiss' method that the problem $((D_n-A_{\mathrm{II}})u=f, u''+Qu'=g)$ has a priori estimate

$$\|(D_n-A_{\mathrm{II}})u\|_{0,\gamma}+\langle\!\langle g \rangle\!\rangle_{1/2,\gamma} \geq C\gamma\,\|u\|_{0,\gamma} \qquad (C>0)$$

where $\mathrm{supp}\, u \subset U(x_0)$, spectrum of u with respect to $x_0, \cdots, x_{n-1}$ $\subset U(\tau_0,\sigma_0)$. Then from the method of the proof of the above estimate and from Lemma 2.6, we obtain a similar estimate in this case. Here we use the fact that the components k of $K''_{\mathrm{I\,II}}$, $K''_{\mathrm{II\,I}}$ has the following form: in some $U(x_0,\tau_0,\sigma_0)$

$$k(x,\zeta,\sigma)=\tilde{k}(x,0,\sigma)+\zeta\tilde{\tilde{k}}(x,0,\sigma)+0(|\zeta|^2),$$
$$|\tilde{k}(x,0,\sigma)|^2 \leq K\,|Q(x,0,\sigma)| \qquad (K>0)$$

which follows from the last assumption of (II), (β).
Furthermore our assumptions are valid for the dual problem and hence a priori estimate for that problem is also obtained. Thus our proof is complete ([6]).

Remark. The conditions (I), (II), (III) are invariant for certain coordinate transformations. Hence Theorem is applicable for problems defined on any smooth $R^1 \times \Omega$.

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