25. On the Bauer Simplexes and the Uniform Algebras

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(Comm. by Kôsaku YOSIDA, M. J. A., Feb. 12, 1974)

1. A Bauer simplex is a simplex whose set of extreme points is closed. We consider in this note when the state space of a uniform algebra is a Bauer simplex (Proposition 2). The result is applied to the tensor product $A_1\hat{\otimes}A_2$ of uniform algebras A_1 and A_2, and we show that all the Gleason parts of at most one of A_1 and A_2 must be trivial if $A_1\hat{\otimes}A_2$ is u.r.m. (i.e. every maximal measure representing a complex homomorphim of $A_1\hat{\otimes}A_2$ is unique).

The author wishes to thank Professor O. Takenouchi for his helpful suggestions.

2. We shall make use of the definitions and the notions of [1]. Let K be a compact convex subset of some locally convex space and let $\mathcal{A}(K)$ denote the Banach space of real valued continuous affine functions on K. The set of extreme points of K is denoted by ∂K.

First we give a slight generalization of Bauer's theorem (cf., [1, p. 105]).

Proposition 1. *Let K be a compact convex set and E a real complete locally convex space. Then K is a Bauer simplex if and only if every continuous map f of ∂K into E has an extension to a continuous affine map of K into E. In particular, if E is a Banach space, then this extension can be made norm preserving.*

Proof. Assume that K is a Bauer simplex. Then ∂K is a closed subset of K. Hence $f(\partial K)$ is a compact subset of E. Since E is complete, the closed convex hull F of $f(\partial K)$ is a compact convex subset. By Bauer's theorem, every boundary measure annihilating $\mathcal{A}(K)$ is null, and from Alfsen [2, Corollary to Theorem A] there exists a continuous affine map $\tilde{f}$ of K into F such that $\tilde{f}|_{\partial K}=f$. If E is a Banach space, then $\|f\|_{\partial K}=\|\tilde{f}\|_K$. The converse statement is reduced to Bauer's theorem by considering a one-dimensional subspace of E. This completes the proof.

3. Let A be a uniform algebra on a compact Hausdorff space X. We denote by $\partial_A X$, $\Gamma(A)$, $M(A)$ and $S(A)$ the Choquet boundary, the Šilov boundary, the maximal ideal space and the state space of A respectively. A is called a Dirichlet algebra if $\mathrm{Re}\, A|_{\Gamma(A)}$ is dense in $C_R(\Gamma(A))$.

For a part of the following Proposition we refer to Fuhr and Phelps [6, Corollary 6.3].

Proposition 2. *Let A be a uniform algebra on a compact Hausdorff space X. Then A is a Dirichlet algebra if and only if $S(A)$ is a Bauer simplex.*

Proof. Suppose that $S(A)$ is a Bauer simplex, then $\mathcal{A}(S(A))|_{\partial S(A)} = C_R(\partial S(A))$ and $\partial_A X = \Gamma(A)$ by Bauer's theorem. Hence $\operatorname{Re} A|_{\Gamma(A)}$ is dense in $C_R(\Gamma(A))$ since $\overline{\operatorname{Re} A}$ is isometrically isomorphic to $\mathcal{A}(S(A))$. The converse is trivial. The proof is complete.

It is immediate from Proposition 2 that $S(A)$ is not a simplex if A is a logmodular algebra which is not Dirichlet (cf., [6, Proposition 6.4]).

4. Let A_1 and A_2 be uniform algebras on compact Hausdorff spaces X_1 and X_2 respectively. We denote by $A_1 \hat{\otimes} A_2$ the uniform closure as a function space on $X_1 \times X_2$ of algebraic tensor product $A_1 \otimes A_2$. Let $\mathcal{BA}(S(A_1) \times S(A_2))$ be the Banach space of continuous biaffine functions on $S(A_1) \times S(A_2)$. The state space of $\mathcal{BA}(S(A_1) \times S(A_2))$ is denoted by $S(A_1) \otimes S(A_2)$. Let Φ denote the natural embedding of $\operatorname{Re}(A_1 \hat{\otimes} A_2)$ into $\mathcal{BA}(S(A_1) \times S(A_2))$:

$$(\Phi(\operatorname{Re}(f \otimes g)))(x, y) = \operatorname{Re}(\langle f, x\rangle \langle g, y\rangle),$$
$$\text{for } f \otimes g \in A_1 \otimes A_2,\ (x, y) \in S(A_1) \times S(A_2),$$

Φ^* be the adjoint map of Φ. Let η be the canonical embedding of $S(A_1) \times S(A_2)$ into $S(A_1 \hat{\otimes} A_2)$:

$$\eta(x, y)(f \otimes g) = \langle f, x\rangle \langle g, y\rangle \qquad \text{for } f \otimes g \in A_1 \otimes A_2,\ (x, y) \in S(A_1) \times S(A_2).$$

Theorem 3. *Suppose that A_1 and A_2 are uniform algebras on compact Hausdorff spaces X_1 and X_2 respectively. Then $S(A_1 \hat{\otimes} A_2)$ is affinely homeomorphic to $S(A_1) \otimes S(A_2)$ if*

(i) *$A_1 \otimes A_2$ is a Dirichlet algebra, or*

(ii) *$A_1 = C(X_1)$ or $A_2 = C(X_2)$.*

Proof. (i) If $A_1 \hat{\otimes} A_2$ is a Dirichlet algebra, A_1 and A_2 are Dirichlet algebras (cf., [9, p. 144]). Hence Proposition 2 and Lazar [7] show that $S(A_1) \otimes S(A_2)$ is a simplex. On the other hand, it follows from Mochizuki [9, Theorem 1] and Namioka and Phelps [10, Theorem 2.3] that $\Phi^*(\partial(S(A_1) \otimes S(A_2))) = \partial S(A_1 \hat{\otimes} A_2)$. Therefore $S(A_1) \otimes S(A_2)$ is a Bauer simplex and $\Phi^*(S(A_1) \otimes S(A_2)) = S(A_1 \hat{\otimes} A_2)$. It remains only to show that Φ^* is one-to-one on $S(A_1) \otimes S(A_2)$. Let z_1 and z_2 be two distinct elements of $S(A_1) \otimes S(A_2)$. Then by Choquet-Meyer's theorem, there is a unique maximal measure μ_i on $S(A_1) \otimes S(A_2)$ which represents z_i $(i=1, 2)$. We have $\mu_1 \neq \mu_2$. Hence there exists a function $f \in C_R(\partial(S(A_1 \hat{\otimes} A_2))$ such that $\int_{\partial S(A_1 \hat{\otimes} A_2)} f\, d\mu_1 \circ \Phi^{*-1} \neq \int_{\partial S(A_1 \hat{\otimes} A_2)} f\, d\mu_2 \circ \Phi^{*-1}$, because $\operatorname{supp}(\mu_i) \subset \partial(S(A_1) \otimes S(A_2))$ $(i=1, 2)$. By hypothesis and Proposition 2, there exists a function $\tilde{f} \in \mathcal{A}(S(A_1 \hat{\otimes} A_2))$ such that $\tilde{f}|_{\partial S(A_1 \hat{\otimes} A_2)} = f$. Hence

$$\tilde{f}(\Phi^*(z_1))=\int \tilde{f}d\mu_1\circ\Phi^{*-1}=\int_{\partial S(A_1\hat{\otimes}A_2)} fd\mu_1\circ\Phi^{*-1}$$
$$\neq\int_{\partial S(A_1\hat{\otimes}A_2)} fd\mu_2\circ\Phi^{*-1}=\int \tilde{f}d\mu_2\circ\Phi^{*-1}=\tilde{f}(\Phi^*(z_2)).$$

Thus Φ^* is one-to-one.

(ii) If $A_1=C(X_1)$, then $\mathrm{Re}\,A_1=C_R(X_1)$ and so $\overline{\mathrm{Re}\,(A_1\hat{\otimes}A_2)}=\overline{\mathrm{Re}\,A_1\otimes\mathrm{Re}\,A_2}$. Since $S(A_1)$ is Bauer simplex, we can prove that $\mathcal{A}(S(A_1))\otimes\mathcal{A}(S(A_2))$ is dense in $\mathcal{B}\mathcal{A}(S(A_1)\times S(A_2))$, (cf., [10]). Hence Φ is an isometric isomorphism. Thus the proof is complete.

The following Corollary is evident.

Corollary. *Let A_1 and A_2 be Dirichlet algebras on compact Hausdorff spaces X_1 and X_2 respectively. Then $A_1\hat{\otimes}A_2$ is a Dirichlet algebra if and only if $S(A_1\hat{\otimes}A_2)$ is affinely homeomorphic to $S(A_1)\otimes S(A_2)$.*

5. Let A be a uniform algebra on a compact Hausdorff space X. Then for $x\in S(A)$, we denote the minimal face which contains x by face (x). It was proved in [1, p. 122] that face $(x)=\bigcup_{\alpha\geqq 1}D_\alpha(x)$, where $D_\alpha(x)=(\alpha x-(\alpha-1)S(A))\cap S(A)$. If we define a relation $\approx$ on $S(A)$ by agreeing that $x\approx y$ if and only if face $(x)=$ face (y), then $\approx$ is an equivalence relation. The equivalence classes of $S(A)$ defined by the relation $\approx$ are called the parts. We notice that

$(*)$ $x\approx y$ if and only if $\sup\{|\log\langle u,x\rangle-\log\langle u,y\rangle| : u\in\mathrm{Re}\,A, u>0\}<\infty$

(cf., [4]). Let A_1 and A_2 be uniform algebras on compact Hausdorff spaces X_1 and X_2 respectively. Then we have the following

Lemma. *Let x_i and y_i be elements of $S(A_i)$ $(i=1,2)$. Then $\eta(x_1,x_2)\approx\eta(y_1,y_2)$ if and only if $x_1\approx y_1$ and $x_2\approx y_2$.*

Proof. If $\eta(x_1,x_2)\approx\eta(y_1,y_2)$, then $x_1\approx y_1$ and $x_2\approx y_2$ since $(*)$ and $u_1\otimes 1$, $1\otimes u_2\in\mathrm{Re}\,(A_1\hat{\otimes}A_2)$ for any $u_1\in\mathrm{Re}\,A_1$ and $u_2\in\mathrm{Re}\,A_2$. For the converse, it is sufficient to prove that face $(\eta(x_1,x_2))=$ face $(\eta(y_1,x_2))$ and face $(\eta(y_1,x_2))=$ face $(\eta(y_1,y_2))$. We note that $D_\alpha(x_1)\supset D_\beta(x_1)$ generally for $\alpha\geqq\beta\geqq 1$. Then for any $z\in$ face $(\eta(x_1,x_2))$, there exist $\alpha\geqq 1$, $z_1\in S(A_1\hat{\otimes}A_2)$ and $x\in S(A_1)$ such that

$$z=\alpha\eta(x_1,x_2)-(\alpha-1)z_1 \quad\text{and}\quad x_1=\alpha y_1-(\alpha-1)x.$$

Hence $z=\alpha^2\eta(y_1,x_2)-(\alpha^2-1)\left\{\frac{\alpha}{\alpha+1}\eta(x,x_2)+\frac{1}{\alpha+1}z_1\right\}$. Since $\frac{\alpha}{\alpha+1}\eta(x,x_2)+\frac{1}{\alpha+1}z_1\in S(A_1\hat{\otimes}A_2)$, we have $z\in$ face $(\eta(y_1,x_2))$. Thus face $(\eta(x_1,x_2))\subset$ face $(\eta(y_1,x_2))$. This completes the proof.

Since for any Gleason part P of $M(A)$ there is a part P_0 of $S(A)$ such that $P=P_0\cap M(A)$, the above Lemma is a generalization of Mochizuki [9, Lemma 3].

We recall that a uniform algebra A is said u.r.m. if for each $x\in M(A)$ there is a unique representing measure for x supported on

$\Gamma(A)$. Then we have the following theorem.

Theorem 4. *If $A_1 \hat{\otimes} A_2$ is u.r.m., then all Gleason parts for at least one of A_1 and A_2 must be trivial.*

Proof. Assume that P_i is a non-trivial Gleason part of $M(A_i)$ $(i=1,2)$. By Lemma, $P=\eta(P_1 \times P_2)$ is a Gleason part of $M(A_1 \hat{\otimes} A_2)$. On the other hand, it follows immediately from the hypothesis and $M(A_1 \hat{\otimes} A_2)=\eta(M(A_1) \times M(A_2))$ [9, Theorem 2] that A_1 and A_2 are u.r.m. Hence by Wermer's embedding theorem, there exist homeomorphisms τ_1, τ_2 and τ of the open unit disk D onto the Gleason parts (P_1, d_1), (P_2, d_2) and (P, d) respectively, where d_1, d_2 and d are the corresponding part metrics. Let $\eta(x_1, x_2)$, $\eta(y_1, y_2)$ be elements of $\eta(P_1 \times P_2)$. Then

$$
\begin{aligned}
&d(\eta(x_1, x_2), \eta(y_1, y_2)) \\
&\quad = \sup \{|\log \langle u, \eta(x_1, x_2)\rangle - \log \langle u, \eta(y_1, y_2)\rangle| : u \in \operatorname{Re}(A_1 \hat{\otimes} A_2), u>0\} \\
&\quad \geqq \sup \{|\log \langle u_1 \otimes 1, \eta(x_1, x_2)\rangle - \log \langle u_1 \otimes 1, \eta(y_1, y_2)\rangle| : u_1 \in \operatorname{Re} A_1, u_1>0\} \\
&\quad = \sup \{|\log \langle u_1, x_1\rangle - \log \langle u_1, y_1\rangle| : u_1 \in \operatorname{Re} A_1, u_1>0\} \\
&\quad = d_1(x_1, y_1).
\end{aligned}
$$

Hence $2d \geqq d_1 + d_2$ and so $(\tau_1^{-1} \times \tau_2^{-1}) \circ \tau$ is a bijective continuous map of D onto $D \times D$. This contradicts the invariance of the dimension. Therefore P_1 or P_2 must be trivial. The proof is complete.

Remark. It has been conjectured that if every Gleason part for A is trivial, then $A=C(X)$. Wilken [13] proved the conjecture for $\mathscr{R}(X)$ when X is a compact set in the plane. If this conjecture is true for any Dirichlet algebra, then we will be able to confirm by Theorem 4 that if $A_1 \otimes A_2$ is Dirichlet, then $A_1=C(X_1)$ or $A_2=C(X_2)$. (This conjecture is known to be false without any assumption on A by the counter examples of Cole [5] and Basener [3].)

References

[1] E. M. Alfsen: Compact Convex Sets and Boundary Integrals. Springer Verlag, Berlin (1971).

[2] ——: On the Dirichlet problem of the Choquet boundary. Acta Math., **120**, 149–159 (1968).

[3] R. F. Basener: An example concerning peak points. Notices Amer. Math. Soc., **18**, 415–416 (1971).

[4] H. S. Bear: Lectures on Gleason Parts. Springer-Verlag, Berlin (1970).

[5] B. Cole: One Point Parts and the Peak Point Conjecture. Ph. D. dissertation, Yale University (1968).

[6] R. Fuhr and R. R. Phelps: Uniqueness of complex representing measures on the Choquet boundary. J. Functional Analysis, **14**, 1–27 (1973).

[7] A. Lazar: Affine products of simplexes. Math. Scand., **22**, 165–175 (1968).

[8] G. Lumer: Analytic functions and Dirichlet problem. Bull. Amer. Math. Soc., **70**, 98–104 (1964).

[9] N. Mochizuki: The tensor product of function algebras. Tohoku Math. J., **17**, 139–146 (1965).

[10] I. Namioka and R. R. Phelps: Tensor products of compact convex sets. Pacific J. Math., **13**, 469–480 (1969).
[11] R. R. Phelps: Lectures on Choquets Theorem. Van Nostrand (1966).
[12] E. L. Stout: The Theory of Uniform Algebras. Bogden & Quigley, Inc. Publishers (1971).
[13] D. R. Wilken: Lebesgue measure of parts for $R(X)$. Proc. Amer. Math. Soc., **18**, 508–512 (1967).