141. *Conformal Sewings of Slit Regions*

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(Comm. by Kinjirô KUNUGI, M. J. A., Oct. 12, 1974)

In this paper we shall consider the Riemann surfaces of *infinite genus* obtained by conformal sewings of slit regions and state some properties induced from the distribution of the slits. This note is an announcement of the results and the details will be given in another paper together with related topics.

1. Let R be a plane region and β be the boundary of R. Take mutually disjoint slits $\{\gamma_n\}_{n=1}^{\infty}$ in R clustering nowhere in R. Denote $G=R-\bigcup_n \gamma_n$. Now we shall define the conformal sewings. (1) Around each γ_j, take a parametric disk U_j so that γ_j is represented as $\{z_j; |\mathrm{Re}\, z_j|\leqq 1, \mathrm{Im}\, z_j=0\}$ by the local parameter z_j, denote by γ_j^+ and γ_j^-, the upper edge and the lower one of γ_j respectively, (2) partition $\{\gamma_n\}$ into finite sections $(\gamma_i)_{i\in I_k}$ $(k=1,2,\cdots)$ and (3) for each arrangement of elements in every finite section $(\gamma_i)_{i\in I_k}$, say $(\gamma_{i_1}, \gamma_{i_2}, \cdots, \gamma_{i_{n(k)}})$, identify γ_j^- with γ_{j+1}^+ $(j=i_1, i_2, \cdots i_{n(k)-1})$ and $\gamma_{i_{n(k)}}^-$ with $\gamma_{i_1}^+$ $(k=1,2,\cdots)$. Then we obtain a Riemann surface $S(G)$ and call such an operation a *conformal sewing* of G.

Definition. We say that slit region G belongs to class O_1 (resp. O_2) if $S(G)\in O_G$ for any (resp. some) conformal sewing of G.

Then we have,

Proposition 1. *There exists a slit region with an infinite number of slits, which belongs to class* O_1.

Now we introduce two families F^1 and F^0 of curves in $G-\overline{G}_0$, where G_0 is a parametric disk in G. F^1 consists of all c such that c is a finite union of closed Jordan curves in $G-\overline{G}_0$ and separates β from ∂G_0. While, $c\in F^0$ iff c is a finite union of closed curves which are closed or join some γ_n with γ_m and c separates β from ∂G_0. We denote by $\lambda(F^1)$ (resp. $\lambda(F^0)$) the extremal length of F^1 (resp. F^0).

Definition. We say that G is of weak (resp. semiweak) type if $\lambda(F^1)=0$ (resp. $\lambda(F^0)=0$). And we say G is of parabolic type if G has no non-constant HB-functions vanishing along $\partial G=\bigcup_n \gamma_n$.

We can prove by extremal length methods the following inclusion relations:

$$\begin{tikzcd}[row sep=small]
O_1 \arrow[dr, shift left] & & & & \\
& O_2 \arrow[ul, shift left, "/" marking] \arrow[dl, shift left, "/" marking] \arrow[r] & \text{semiweak} \arrow[r, shift left] & \hat{G} \in O_G \arrow[l, shift left] \arrow[r, shift left] & \text{parabolic} \arrow[l, shift left, "/" marking] \\
\text{weak} \arrow[uu, "/" marking] \arrow[ur, shift left] & & & &
\end{tikzcd}$$

where $\hat{G}$ stands for the *double* of G along ∂G.

Next, we consider the case where β is a compact set on the real axis and γ_n are contained in the upper half plane. Moreover we identify the sides of γ_n with those of $\bar{\gamma}_n=\{\bar{z}\,;\,z\in\gamma_n\}$ symmetrically, i.e. in the manner folding w.r.t. the real axis to obtain $S(G)$. Let O' be the class of Riemann surfaces defined in Kusunoki [1]. Then we can prove,

Proposition 2. *G is of weak (resp. semiweak) type iff $S(G)\in O'$ (resp. O_G).*

2. Here we shall give counter examples corresponding to '$\not\to$' in the schema and note some properties they have.

(A) Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of strictly increasing positive numbers such that $a_n\to\infty$. Let $\beta=\{\infty\}$ and $\gamma_n=\{z\,;\,|\mathrm{Re}\,z|\leqq 1, \mathrm{Im}\,z=a_n\}$. Identify γ_{2n} with γ_{2n+1} symmetrically w.r.t. the line $y=(a_{2n}+a_{2n+1})/2$ so that we get $S(G)$. Suppose $\sum_{n=1}^{\infty}(a_{2n}-a_{2n-1})<\infty$, and $a_{2n+1}/a_{2n}\geqq n$ ($n=1,2,\cdots$). Then $S(G)\notin O_G$, while G is of weak type.

(B) We consider the radial slit disk $G=\boldsymbol{F}$ and the Riemann surface $S(G)=\hat{\boldsymbol{F}}\in O_{HB}-O_G$ obtained by a conformal sewing of G (cf. Tôki [2]). This $G=\boldsymbol{F}$ will give an example showing that 'parabolic$\not\to$semiweak'.

We can prove by using $\boldsymbol{F}$ and $\hat{\boldsymbol{F}}$, that *there exists a region of type NO_{HB}* (i.e. on which there exist no non-constant HB-functions with vanishing normal derivatives along the relative boundary) *which has a subregion not belonging to NO_{HB}.* (Note that every subregion of a region $\in SO_{HB}$ is of type SO_{HB}.)

(C) Let $\{b_n\}$ be a strictly decreasing sequence of positive numbers converging to 0. Let $\beta=\{z\,;\,|\mathrm{Re}\,z|\leqq 1, \mathrm{Im}\,z=0\}$ and $\gamma_n=\{z\,;\,|\mathrm{Re}\,z|\leqq 1, \mathrm{Im}\,z=a_n\}$. Identify γ_n with $\bar{\gamma}_n$ symmetrically w.r.t. the real axis to obtain $S(G)$ (cf. Proposition 2). We can prove that '$O_2\not\to O_1$' and '$O_2\not\to$weak'.

The ideal boundary $\varDelta_M$ of the Martin compactification of above $S(G)$ becomes the fibre over $[-1,1]$ and the minimal points lie only over two points $1,-1$. By this example we know that there exists a non-constant AB-function which converges to zero uniformly on curves $\{C_n\}$ tending to non-degenerate Martin boundary $\varDelta_M$. This shows that *the extenison of classical Koebe's theorem to open Riemann surfaces with Martin compactification does not hold under only assumption that Koebe sequence $\{C_n\}$ converge to a continuum.*

Furthermore, from (C) we can prove that for any $M>0$ there exists a hyperbolic Riemann surface S such that the level curves $G(p)=r$ $(0<r<M)$ of Green function G of S are *all non-compact.*

References

[1] Kusunoki, Y.: Some classes of Riemann surfaces characterized by the extremal length. Proc. Japan Acad., **32**, 406–408 (1956).

[2] Tôki, Y.: On the examples in the classification of open Riemann surfaces. I. Osaka Math. J., **5**, 267–280 (1953).