131. On a Theorem of Wallace and Tsushima

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1. As was pointed out in Math. Reviews 22 (1961), #12146, the proof of [8; Theorem] contains an error but the theorem holds good for solvable groups and groups with p-complement. Recently, Y. Tsushima [6] has showed that the theorem is still true for p-solvable groups. In the present paper, we shall give an alternative proof to the above fact, and several related results on the radical of a group algebra. We are indebted to Mr. Y. Ninomiya and Mr. Y. Tsushima for their useful advice.

Let K be an algebraically closed field of characteristic $p>0$, and G a finite group with a normal subgroup N such that $|N|$ is not a power of p and G/N is a p-group. Further, G_p will represent a p-Sylow subgroup of G, KG the group algebra of G over K, $J(KG)$ the radical of KG, and $[J(KG):K]$ the K-dimension of $J(KG)$.

2. Now, let $\{T_1, T_2, \cdots, T_s\}$ be the set of all non-conjugate irreducible KN-modules, and G_i the inertia group of T_i, where T_1 corresponds to the 1-representation. By [4; (III. 3.1)] each T_i can be extended uniquely to an irreducible module $\hat{T}_i$ of G_i. We shall prove first the following:

Lemma 1 (cf. [2, (50.2)]). *$\{\hat{T}_1^G, \hat{T}_2^G, \cdots, \hat{T}_s^G\}$ is the set of all irreducible modules of G.*

Proof. At first, we shall show that $\hat{T}_i^G$ is irreducible (cf. [5, Lemma 2]). Let M be a maximal KG-submodule of $\hat{T}_i^G$. By $\mathrm{Hom}_{KG_i}(\hat{T}_i, \hat{T}_i^G/M) \cong \mathrm{Hom}_{KG}(\hat{T}_i^G, \hat{T}_i^G/M) \neq 0$, there exists a KG_i-submodule S_i of $\hat{T}_i^G/M$, which is KG_i-isomorphic to $\hat{T}_i$. By Clliford's theorem, $\hat{T}_i^G/M$ is KN-isomorphic to a direct sum of e-copies of $\sum_{r=1}^{t} \oplus T_i^{(x_r)}$, where $\{x_r\}$ is a left cross section of G_i in G. Therefore, $(G:G_i)[T_i:K]=[\hat{T}_i^G:K] \geqq [\hat{T}_i^G/M:K]=e(G:G_i)[T_i:K]$ and $[\hat{T}_i^G:K]=[\hat{T}_i^G/M:K]$, which means that $M=0$ and $\hat{T}_i^G$ is irreducible. Next, we shall prove that the above modules are all non-isomorphic. Let $\{y_l \mid 1 \leqq l \leqq r\}$ is a left cross section of G_j in G. Then $\mathrm{Hom}_{KG}(\hat{T}_i^G, \hat{T}_j^G) \cong \mathrm{Hom}_{KG_i}(\hat{T}_i, \hat{T}_j^G) \subseteq \mathrm{Hom}_{KN}(T_i, \hat{T}_j^G) \cong \sum_{l=1}^{r} \oplus \mathrm{Hom}_{KN}(T_i, y_l \otimes T_j)=0$ for $i \neq j$. Hence, it remains only to prove that s is the number of p-regular classes of G. Let $\{S_1, S_2, \cdots, S_k\}$ be the set of all irreducible representations of N, ω_i Brauer character of S_i. Then ω_i is conjugate to ω_j if and only if S_i is conjugate to S_j. By Brauer's permutation lemma [3, (12.1)], the number of orbits of a permutation group G/N acting on $\{S_1, S_2, \cdots, S_k\}$ is the number of orbits of a permutation group G/N acting on the set of p-regular classes of N. Noting that every p-regular element of G is contained in N, we can see that s is the number of p-regular classes of G.

Concerning the K-dimension of $J(KG)$, we obtain the following

Theorem 2. (1) $[J(KG): K]=|G|-\sum_{i=1}^{s}(G: G_i)^2[T_i: K]^2$.

(2) $|G|-|N|+[J(KN): K]\geqq[J(KG): K]\geqq(G: N)([J(KN): K]+1)-1$.

Proof. (1) is evident by Lemma 1.

(2) $[J(KG): K]=|G|-\sum_{i=1}^{s}(G: G_i)^2[T_i: K]^2\leqq|G|-\sum_{i=1}^{s}(G: G_i)[T_i: K]^2=|G|-|N|+[J(KN): K]$, which proves the first inequality. Next, we shall show the second one. $[J(KG): K]=|G|-\sum_{i=1}^{s}(G: G_i)^2[T_i: K]^2\geqq|G|-1-(G: N)\sum_{i=2}^{s}(G: G_i)[T_i: K]^2=|G|-1-(G: N)(|N|-[J(KN): K]-1)=(G: N)([J(KN): K]+1)-1$.

3. A pair (G,N) is called large (resp. small) if and only if $[J(KG):K]=|G|-|N|+[J(KN):K]$ (resp. $[J(KG):K]=(G:N)([J(KN):K]+1)-1$). These conditions are characterized as follows:

Theorem 3. (1) *(G,N) is large if and only if $G=N\cdot C_G(a)$ for every p-regular element a of N.*

(2) *(G,N) is small if and only if $C_G(a)\subseteq N$ for every p-regular element a of $N-1$.*

Proof. (1) If (G,N) is large, then $G_i=G$ for every i. By Brauer's permutation lemma [3, (12.1)], $\mathfrak{C}^h=\mathfrak{C}$ for every p-regular class $\mathfrak{C}$ of N and every element h of G. Hence, $(G: C_G(a))=(N; C_N(a))$ and $G=N\cdot C_G(a)$ for $a\in\mathfrak{C}$. Conversely, if $G=N\cdot C_G(a)$ and $\mathfrak{C}$ is a conjugate class of N containing a p-regular element a, then $\mathfrak{C}^h=\mathfrak{C}$ for every element h of G. Thus, by Brauer's permutation lemma, $G_i=G$ for every i and (G,N) is large by Theorem 2(1).

(2) If (G,N) is small, then $G_i=N$ for $i\neq 1$. By Brauer's permutation lemma, $\mathfrak{C}^h\neq\mathfrak{C}$ for every p-regular class $\mathfrak{C}\neq 1$ of N and $h\in G-N$. Thus, $(G: C_G(a))=|\bigcup_{h\in G}\mathfrak{C}^h|=(G: N)\cdot(N: C_N(a))$ and $C_G(a)=C_N(a)\subseteq N$ for $a\in\mathfrak{C}$. Conversely, assume that $C_G(a)\subseteq N$ for every p-regular element $a\neq 1$ of N. If $\mathfrak{C}$ is the conjugate class of N containing a, then $|\bigcup_{h\in G}\mathfrak{C}^h|=(G: C_G(a))=(G: N)(N: C_N(a))$ and $\mathfrak{C}^h\neq\mathfrak{C}$ for $h\in G-N$. Thus, by Brauer's permutation lemma, $G_i=N$ for $i\neq 1$ and (G,N) is small by Theorem 2(1).

Next Theorem 4 is fundamental in this paper.

Theorem 4. *Assume that N is a p'-group.*

(1) *(G,N) is large if and only if G is a direct product of N and G_p*

(2) *(G,N) is small if and only if G is a Frobenius group with kernel N and complement G_p.*

Proof. (1) If G is a direct product of N and G_p, then the assertion is trivial by Theorem 2(1). Conversely, if (G,N) is large then by

Theorem 3(1), $G=N\cdot C_G(a)$ for all $a\in N$ and $C_G(a)/C_N(a)\cong G/N\cong G_p$. Thus, $C_G(a)=C_N(a)\cdot xG_px^{-1}$, and we may assume $x\in N$ by $G=N\cdot G_p$. Hence, a is contained in $xC_G(G_p)x^{-1}\cap N=xC_N(G_p)x^{-1}$. Accordingly, $N=\bigcup_{i=1}^{l} x_iC_N(G_p)x_i^{-1}$, where $\{x_1G_px_1^{-1}=G_p, x_2G_px_2^{-1}, \cdots, x_lG_px_l^{-1}\}$ is the set of all p-Sylow subgroups of G and x_i are elements of N. Thus, we can see that $N=x_1C_N(G_p)x_1^{-1}=C_N(G_p)$ by $N=\bigcup_{i=1}^{l} x_iC_N(G_p)x_i^{-1}$ and hence G is a direct product of N and G_p.

(2) is a direct consequence of Theorem 3(2) (cf. the proof of [7, Theorem 2]).

The proof of the following is immediate, and will be omitted.

Lemma 5. *B is a normal p-subgroup of a finite group A, then* $[J(KA):K]=|A|-|A/B|+[J(K(A/B)):K]$.

As a combination of Theorem 4 and Lemma 5, we obtain the next:

Corollary 6. *Assume that a p-Sylow subgroup N_p of N is normal.*

(1) *(G,N) is large if and only if G/N_p is a direct product of N/N_p and G_p/N_p (i.e. G_p is normal in G).*

(2) *(G,N) is small if and only if G/N_p is a Frobenius group with kernel N/N_p and complement G_p/N_p.*

Next Lemma is trivial

Lemma 7. *Let $A\supseteq B\supseteq C$ be a series of normal subgroups of a finite group A such that A/B and B/C are p-groups. Then (A,C) is large (resp. small) if and only if both (A,B) and (B,C) are large (resp. small).*

The following is a combination of Lemma 7 and Theorem 4(2).

Corollary 8. *Let N be a Frobenius group with kernel $N_{p'}$ and complemet N_p. Then, (G,N) is small if and only if G is a Frobenius group with kernel $N_{p'}$ and complement G_p.*

4. In this section, we shall give an alternative proof to the fact indicated by Y. Tsushima ([6, Theorem 2]).

Theorem 9. *Let H be a p-solvable group, and u_1 the degree of principal indecomposable module U_1 such that $U_1/J(U_1)$ is a trivial module. Then, $[J(KH):K]=|H|-|H|/u_1$ if and only if H_p is normal in H.*

Proof. If H_p is normal, then $u_1=|H_p|$ (cf. [1, p. 583]) and $[J(KH):K]=(H:H_p)(|H_p|-1)=|H|-|H|/u_1$. Conversely, assume that $[J(KH):K]=|H|-|H|/u_1$. We shall proceed by induction concerning the order of H. Since H is p-solvable, there exists a normal subgroup $\tilde{H}$ such that $p=(H:\tilde{H})$ or $p\nmid(H:\tilde{H})$. We shall distinguish between two cases.

Case 1. $p\nmid(H:H)$; $u_1\geqq\tilde{u}_1$ by Fobenius reciprocity law (cf. [1, p. 583]), where $\tilde{u}_1$ is defined for $\tilde{H}$ as similar as u_1 was done for H. Thus, by [1, p. 580], $[J(K\tilde{H}):K]\leqq|\tilde{H}|-|\tilde{H}|/\tilde{u}_1\leqq|\tilde{H}|-|\tilde{H}|/u_1=[J(KH):K]/(H:\tilde{H})=[J(K\tilde{H}):K]$ and hence $[J(K\tilde{H}):K]=|\tilde{H}|-|\tilde{H}|/u_1$. Therefore,

$\tilde{H}_p$ is normal in $\tilde{H}$ by induction hypothesis. Since $\tilde{H}_p$ is characteristic in $\tilde{H}$, $H_p=\tilde{H}_p$ is normal in H.

Case 2. $p=(H:\tilde{H})$: By Theorem 2(2) and $u_1=p\cdot\tilde{u}_1$ (cf. [4, (III. 3.8]), $[J(KH):K]\leqq|H|-|\tilde{H}|+[J(K\tilde{H}):K]\leqq|H|-|\tilde{H}|+|\tilde{H}|-|\tilde{H}|/\tilde{u}_1=|H|-|H|/u_1=[J(KH):K]$. Hence, $[J(K\tilde{H}):K]=|\tilde{H}|-|\tilde{H}|/u_1$ and $(H,\tilde{H})$ is large. Thus, $\tilde{H}_p$ is normal in $\tilde{H}$ and hence H_p is normal in H by Corollary 6(1).

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