166. *Note on Approximation of Nonlinear Semigroups*

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Let X be a Banach space with a norm $|\ \ |$, and let $\{T(t); t\geqq 0\}$ be a *contraction* (*nonlinear*) *semigroup* on a closed *convex* subset C of X, namely a family of operators from C into C satisfying the following conditions:

(i) $T(0)=I$ (the identity), $T(t+s)=T(t)T(s)$ for $t, s\geqq 0$;

(ii) $|T(t)x-T(t)y|\leqq|x-y|$ for $t\geqq 0$ and $x, y\in C$;

(iii) $\lim_{t\downarrow 0} T(t)x=x$ for $x\in C$.

For each $\lambda, h>0$ we define

$$A_h=h^{-1}(T(h)-I) \quad \text{and} \quad J_{\lambda,h}=(I-\lambda A_h)^{-1}.$$

It is well known that $J_{\lambda,h}$ is a contraction operator from C into C and that A_h generates a unique contraction semigroup $\{T_h(t); t\geqq 0\}$ on C such that $(d/dt)T_h(t)x=A_hT_h(t)x$ for $x\in C$ and $t\geqq 0$ (e.g. see [1] and [3]). The purpose of the present note is to prove the following

Theorem. *For each* $x\in C$, *we have*

(a) $T(t)x=\lim_{h\downarrow 0} T_h(t)x$

uniformly on every bounded interval of $[0,\infty)$,

(b) $T(t)x=\lim_{h\downarrow 0}\{(1-t)I+tT(h)\}^{[1/h]}x$

uniformly in $t\in[0,1]$, *and*

(c) $T(t)x=\lim_{(\lambda,h)\to(0,0)}(I-\lambda A_h)^{-[t/\lambda]}x$

uniformly on every bounded interval of $[0,\infty)$, *where* $[\ \ ]$ *denotes the Gaussian bracket.*

Remark. These results were obtained for $x\in\bar{E}$ by I. Miyadera [3], where $E=\{x\in C; |A_hx|=O(1) \text{ as } h\downarrow 0\}$. Recently Y. Kobayashi [2] showed that (a) holds true for $x\in C$ by using an advanced convergence theorem.

We now set for $t>0$ and $x\in C$

$$\gamma(t)=8\cdot\sup\{|T(\eta)x-x|; 0\leqq\eta\leqq t\}.$$

Clearly $\gamma(t)$ is non-decreasing and $\gamma(t)\downarrow 0$ as $t\downarrow 0$ by (iii). The following lemma is in Crandall-Liggett [1; Lemma 3.3].

Lemma. *For* $x\in C$ *and* $\delta>0$

$$|J_{\lambda,h}x-x|\leqq\gamma(2\delta) \qquad \textit{if } \lambda, h<\delta.$$

To prove Theorem we start from the following inequalities which are found in [3; (3.4), (3.6) and one in p. 257]:

$$(1)\qquad |T_h(t)x-T([t/h]h)x|\leqq(\sqrt{th}+h)|A_hx|$$
for $x\in C$ and $t\geqq 0$;
$$(2)\qquad |T_h(t)x-\{(1-t)I+tT(h)\}^{[1/h]}x|\leqq(\sqrt{h}+h)\,|A_hx|$$
for $x\in C$ and $t\in[0,1]$;
$$(3)\qquad |T_h(t)x-(I-\lambda A_h)^{-[t/\lambda]}x|\leqq 2\sqrt{\lambda^2+t\lambda}\,|A_hx|$$
for $x\in C$ and $t\geqq 0$.
In what follows we let $x\in C$, $T>0$, $\delta\in(0,1]$ and $\lambda, h<\delta^2$. Since $A_hJ_{\lambda,h}=\lambda^{-1}(J_{\lambda,h}-I)$, it follows from (1), (2), (3) and Lemma that
$$(4)\qquad \begin{aligned}&|T_h(t)J_{\sqrt{h},h}x-T([t/h]h)J_{\sqrt{h},h}x|\\&\quad\leqq(\sqrt{t}+\sqrt{h})\,|J_{\sqrt{h},h}x-x|\leqq(\sqrt{t}+1)\gamma(2\delta)\qquad\text{for } t\geqq 0,\end{aligned}$$
$$(5)\qquad \begin{aligned}&|T_h(t)J_{\sqrt{h},h}x-\{(1-t)I+tT(h)\}^{[1/h]}J_{\sqrt{h},h}x|\\&\quad\leqq(1+\sqrt{h})\,|J_{\sqrt{h},h}x-x|\leqq 2\gamma(2\delta)\qquad\text{for } t\in[0,1]\end{aligned}$$
and
$$(6)\qquad \begin{aligned}&|T_h(t)J_{\sqrt{\lambda},h}x-(I-\lambda A)^{-[t/\lambda]}J_{\sqrt{\lambda},h}x|\\&\quad\leqq 2\sqrt{\lambda+t}\,|J_{\sqrt{\lambda},h}x-x|\leqq 2\sqrt{1+t}\,\gamma(2\delta)\qquad\text{for } t\geqq 0.\end{aligned}$$

Proof of (a). Since
$$\begin{aligned}&|T(t)x-T_h(t)x|\\&\quad\leqq|T(t)x-T([t/h]h)x|+|T([t/h]h)J_{\sqrt{h},h}x-T([t/h]h)x|\\&\qquad+|T([t/h]h)J_{\sqrt{h},h}x-T_h(t)J_{\sqrt{h},h}x|+|T_h(t)J_{\sqrt{h},h}x-T_h(t)x|\\&\quad\leqq|T(t-[t/h]h)x-x|+2|J_{\sqrt{h},h}x-x|+|T([t/h]h)J_{\sqrt{h},h}x-T_h(t)J_{\sqrt{h},h}x|\\&\quad\leqq\gamma(h)+2\gamma(2\delta)+(\sqrt{t}+1)\gamma(2\delta)\end{aligned}$$
$$(7)\qquad \leqq(\sqrt{t}+4)\gamma(2\delta),$$
we get $\sup_{t\in[0,T]}|T(t)x-T_h(t)x|\leqq(\sqrt{T}+4)\gamma(2\delta)$. For any $\varepsilon>0$ taking $\delta\in(0,1]$ so that $\gamma(2\delta)<\varepsilon/(\sqrt{T}+4)$, we have
$$\sup_{t\in[0,T]}|T(t)x-T_h(t)x|<\varepsilon\qquad\text{for } h<\delta^2.$$

Proof of (b). By (5) and (7) we obtain that
$$\begin{aligned}&\sup_{t\in[0,1]}|T(t)x-\{(1-t)I+tT(h)\}^{[1/h]}x|\\&\quad\leqq(1+4)\gamma(2\delta)+\sup_{t\in[0,1]}|T_h(t)x-\{(1-t)I+tT(h)\}^{[1/h]}x|\\&\quad\leqq 5\gamma(2\delta)+2\gamma(2\delta)+2\,|J_{\sqrt{h},h}x-x|\\&\quad\leqq 9\gamma(2\delta),\text{ which implies that (b) is true.}\end{aligned}$$
Here we have used the fact that $(1-t)I+tT(h)$ is a contraction.

Proof of (c). It also follows from (6) and (7) that
$$\begin{aligned}&\sup_{t\in[0,T]}|T(t)x-(I-\lambda A_h)^{-[t/\lambda]}x|\\&\quad\leqq(\sqrt{T}+4)\gamma(2\delta)+\sup_{t\in[0,T]}|T(t)x-(I-\lambda A_h)^{-[t/\lambda]}x|\\&\quad\leqq(\sqrt{T}+4)\gamma(2\delta)+2\sqrt{1+T}\,\gamma(2\delta)+2\,|J_{\sqrt{\lambda},h}x-x|\\&\quad\leqq 3(\sqrt{1+T}+2)\gamma(2\delta).\end{aligned}$$
Hence we complete the proof.

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same argument as (a) and (b). The author wants to express his deep gratitude to Professor I. Miyadera for many valuable advices.

References

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