57. *On Involutive Systems of Partial Differential Equations whose Characters of Order more than One Vanish*

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0. Introduction. The structure of involutive systems of partial differential equations with one unknown function of *two* independent variables was recently investigated in detail and Darboux's method of integration was extended to those systems by the author [6], [7]. Our main aim is firstly to investigate the structure of involutive systems of partial differential equations with one unknown function of *several* independent variables whose characters of order more than one vanish and secondly to obtain a method of integration similar to Darboux's method for such systems. Although some of our results are derived from E. Cartan's ones on involutive differential systems whose characters of order more than one vanish (Cartan [3], §§ 97–100), our results are much more complete than E. Cartan's. The descriptions of our results are similar to those in the case of two independent variables obtained by the author himself [7]. However the arguments concerning algebraic considerations are pretty different from those in the memoir [7]. All notions which appear in this note are assumed to be in the category of real or complex analyticity although all arguments except when the existence theorem of Cartan-Kähler is applied can be done in the category of differentiability. Details of this note will be published elsewhere.

1. Involutive systems. Let Φ be a system of partial differential equations of order m with one unknown function. Φ is a system defined in $J^m(M,N,\rho)$, the space of m-jets of sections of a fibered manifold (M,N,ρ) in which $\dim M=\dim N+1$. Let $(x_1,\cdots,x_n,z)$ be a local coordinate system of M ($n=\dim N$) associated with (M,N,ρ). Let $p_{i_1\cdots i_l}(j^m_a(f))$ denote $\partial^l z(x)/\partial x_{i_1}\cdots\partial x_{i_l}(a)$, where $z(x_1,\cdots,x_n)$ denotes z-coordinate of the section f around $a\in N$. A local coordinate system of $J^m(M,N,\rho)$ is given by

$$(x_1,\cdots,x_n,z,p_{i_1\cdots i_l};\ 1\leqq i_1,\cdots,i_l\leqq n,\quad 1\leqq l\leqq m).$$

Let X be a point of $I\Phi$, the set of integral points of Φ. We shall denote $r_m(X)=\dim\langle\pi^*_m dF;\ F\in\Phi_X\rangle$, $r_{m+1}(X)=\dim\langle\pi^*_{m+1}dF;\ F\in(p\Phi)_{\tilde X}\rangle$, where $\pi^*_m dF$ denotes $\sum_{i_1\leqq\cdots\leqq i_m}\partial F/\partial p_{i_1\cdots i_m}dp_{i_1\cdots i_m}\in T^*_X(J^m)$, $\langle v_\lambda;\ \lambda\in\Lambda\rangle$ denotes the vector space spanned by $\{v_\lambda;\ \lambda\in\Lambda\}$, $p\Phi$ is the (total) prolongation of Φ

and $\tilde{X}$ is an $(m+1)$-jet over X. Let ρ_k^m be the natural projection from $J^m(M, N, \rho)$ to $J^k(M, N, \rho)$ and $Q_X(J^m)$ be ker $d\rho_{m-1}^m$. The space $C_X(\Phi)$ is defined to be $\{\mathfrak{X} \in Q_X(J^m); \mathfrak{X}(\varphi)=0$ for every $\varphi \in \Phi_X\}$. $C_X(\Phi)$ is considered canonically as a subspace of $Q_{\bar{X}}(J^{m-1})\otimes T_a^*(N)$, where $\bar{X}=\rho_{m-1}^m X$, $a=\rho\circ\rho_0^m X$. For a system of vectors $v_1, \cdots, v_k$ in $T_a(N)$, the space $C_X(\Phi)(v_1, \cdots, v_k)$ is defined to be the set of all elements of $C_X(\Phi)$ which annihilate $v_1, \cdots, v_k$. We shall denote $\min\{\dim C_X(\Phi)(v_1, \cdots, v_k); v_1, \cdots, v_k \in T_a(N)\}$ by $g_k(X)$ $(k=1, 2, \cdots, n)$ and $\dim C_X(\Phi)$ by $g_0(X)$. When Φ is involutive at X, the characters of order $1, 2, \cdots, n$ of Φ at X are defined by $s_1(X)=g_0(X)-g_1(X)$, $s_2(X)=g_1(X)-g_2(X), \cdots, s_n(X)=g_{n-1}(X)-g_n(X)$. The meaning of the characters consists in the second existence theorem of Cartan-Kähler. The vanishing of $s_2(X), \cdots, s_n(X)$ is equivalent to the condition that $g_1(X)$ vanishes. In this case $s_1(X)$ is equal to $\dim C_X(\Phi)$. Under this condition, the criterion of involution given by M. Kuranishi [8] and M. Matsuda [9] is stated as follows.

Theorem 1. *Let X_0 be an integral point of Φ. The system Φ is involutive at X_0 and its characters $s_2(X_0), \cdots, s_n(X_0)$ vanish if and only if the following conditions are satisfied:*

(i) *X_0 is an ordinary integral point and $g_1(X_0)$ vanishes.*

(ii) *$r_m(X)$ remains constant on a neighbourhood of X_0 in $I\Phi$.*

(iii) *$r_{m+1}(X)=r_m(X)+C_{n-2}^{m+n-1}$ on a neighbourhood of X_0 in $I\Phi$.*

(iv) *Φ is p-closed at X_0.*

2. Characteristic ideal. We define the *characteristic ideal* of Φ at $X\in I\Phi$ to be that smallest ideal $\mathfrak{M}$ in the ring of polynomials in $\partial/\partial x_1, \cdots, \partial/\partial x_n$ which has the property that $(\partial/\partial x_1)P, \cdots, (\partial/\partial x_n)P\equiv 0$ $(\mathfrak{M})$ implies $P\equiv 0$ $(\mathfrak{M})$ and which contains the polynomials

$$\sum_{i_1\leq\cdots\leq i_m} \partial F(X)/\partial p_{i_1\cdots i_m}(\partial/\partial x_{i_1})\cdots(\partial/\partial x_{i_m}) \in S^m(T_a(N)) \ (F\in\Phi_X).$$

More precisely, when we write $\xi_i=\partial/\partial x_i$, the characteristic ideal $\mathfrak{M}$ is a homogeneous ideal in the polynomial ring $\boldsymbol{R}[\xi_1, \cdots, \xi_n]$ or $\boldsymbol{C}[\xi_1, \cdots, \xi_n]$ according as the argument is done in the category of real or complex analyticity. In the first case, if $\mathfrak{M}$ has "imaginary" zeros, we shall regard in the following $\mathfrak{M}$ as an ideal in $\boldsymbol{C}[\xi_1, \cdots, \xi_n]$. By the well-known theorem, $\mathfrak{M}$ admits an irredundant primary representation by greatest primary components $\mathfrak{M}=\mathfrak{Q}_1\cap\cdots\cap\mathfrak{Q}_\nu$, where $\mathfrak{Q}_i$ are primary homogeneous ideals. This representation is unique in the sense that the number ν of the components and the set of prime (homogeneous) ideals $\mathfrak{P}_i$ belonging to $\mathfrak{Q}_i$ are uniquely determined only by $\mathfrak{M}$ (cf. van der Waerden [10], Zariski and Samuel [11]). Suppose that Φ is involutive at X and that $g_1(X)=0$. Then we can prove that $\mathfrak{M}$ defines an algebraic projective variety of dimension zero, and hence each $\mathfrak{P}_i$ is a homogeneous ideal of a point belonging to that variety. The exponent of $\mathfrak{Q}_i$ is defined to be $\min\{\sigma; \mathfrak{P}_i^\sigma\equiv 0\ (\mathfrak{Q}_i)\}$ $(1\leq i\leq\nu)$. By applying M. Noether's Theorem (van der Waerden [10], § 96) and using condition (iii) in Theorem 1, we can show the following important theorem.

Theorem 2. *Suppose that Φ is involutive at X and that the characters of Φ of order more than one at X vanish. Let $\varepsilon_1, \cdots, \varepsilon_\nu$ denote the exponents of components of the irredundant primary representation of the characteristic ideal of Φ at X, where ν is the number of the components. Then $\sum_{i=1}^{\nu} \varepsilon_i$ is equal to or less than the character $s_1(X)$. If all ε_i are equal to one, $\nu=\sum_{i=1}^{\nu} \varepsilon_i$ is equal to $s_1(X)$.*

Remark. The equality in the first assertion in Theorem 2 is not valid in general except when $n=2$.

3. Characteristic systems and their invariants. Hereafter we shall always assume that Φ satisfies the following two conditions:

(H_1) Φ is involutive at X_0 and the characters $s_2, \cdots, s_n$ at X_0 vanish,

(H_2) Each component of the irredundant primary representation of the characteristic ideal $\mathfrak{M}$ of Φ at X_0 has the exponent one.

It is to be noticed that when (H_1) and (H_2) are satisfied, they are also satisfied at each integral point sufficiently near X_0. In this case each primary component $\mathfrak{Q}_i$ of the irredundant primary representation of $\mathfrak{M}$ coincides with the prime ideal $\mathfrak{P}_i$ belonging to $\mathfrak{Q}_i$ $(1\leqq i\leqq\nu)$, and hence $\mathfrak{M}=\mathfrak{P}_1\cap\cdots\cap\mathfrak{P}_\nu$. Moreover Theorem 2 implies that $\nu=s_1(X_0)$.

Characteristic systems (in the sense of Monge) of Φ around X_0 are defined to be the Pfaffian systems determining singular elements of the following differential system:

$$\Sigma(\Phi)\begin{cases} F=0,\ dF=0 \quad (F \text{ varies all sections of } \Phi \text{ around } X_0), \\ dp_{i_1\cdots i_l}-\sum_{i=1}^{n} p_{i_1\cdots i_l i}dx_i=0,\ \sum_{i=1}^{n} dp_{i_1\cdots i_l i}\wedge dx_i=0 \end{cases}$$

$$(1\leqq i, i_1, \cdots, i_l\leqq n, 0\leqq l<m)$$

where $p_{i_1\cdots i_l}$ denotes z when $l=0$. Remark that $\Sigma(\Phi)$ is involutive at X_0 with respect to N (cf. résumé in Kakié [7], § 1). Characteristic systems are defined corresponding to each component $\mathfrak{P}_i$ and we shall denote by $\mathrm{D}^m(\mathfrak{P}_i)$ the characteristic system corresponding to $\mathfrak{P}_i$.

Let X be an integral point. A function u of m-jets is called a *(relative) invariant* of $\mathrm{D}^m(\mathfrak{P}_i)$ at X if $(u(X)=0,\ du(X)\neq 0$ and if) du vanishes in consequence of $\mathrm{D}^m(\mathfrak{P}_i)$ (and u) on a neighbourhood of X in $I\Phi$. Characteristic systems and invariants of higher order are defined in the usual manner by prolonging $\Sigma(\Phi)$. We say that a function u of m-jets is *independent of Φ at X* if $du\not\equiv 0\ (\mathrm{mod}\ \pi_m^*F;\ F\in\Phi_X)$. Let $u_1, \cdots, u_q$ be functions on an open set $\mathcal{U}$ in $J^m(M, N, \rho)$. We denote by $\Phi^m(u_1, \cdots, u_q)$ the system of partial differential equations on $\mathcal{U}$ generated by Φ and $u_1, \cdots, u_q$. The importance of the invariants consists in the following theorems.

Theorem 3. *Suppose that Φ satisfies* (H_1) *and* (H_2). *Let u be a function of m-jets defined around X_0 which is independent of Φ at X_0. Then $\Phi^m(u)$ is involutive at X_0 if and only if u is a relative invariant of a characteristic system* $\mathrm{D}^m(\mathfrak{P}_j)$. *In this case the characteristic ideal of $\Phi^m(u)$ at X_0 is obtained from $\mathfrak{M}=\mathfrak{P}_1\cap\cdots\cap\mathfrak{P}_\nu$ by omitting the component $\mathfrak{P}_j$* ($\nu=s_1(X_0)$).

Theorem 4. *Suppose that Φ satisfies* (H_1) *and* (H_2). *Let $u_1, \cdots, u_q$ be q relative invariants at X_0 of distinct characteristic systems* $\mathrm{D}^m(\mathfrak{P}_{j_1}), \cdots, \mathrm{D}^m(\mathfrak{P}_{j_q})$ *each of which is independent of Φ at X_0. Then $\Phi^m(u_1, \cdots, u_q)$ is involutive at X_0. In this case the characteristic ideal of $\Phi^m(u_1, \cdots, u_q)$ is obtained from $\mathfrak{M}=\mathfrak{P}_1\cap\cdots\cap\mathfrak{P}_\nu$ by omitting q components $\mathfrak{P}_{j_1}, \cdots, \mathfrak{P}_{j_q}$.*

These theorems are generalized to the case where invariants of higher order occur.

4. A method of integration. Since Φ satisfies (H_1) by assumption, it is reasonable to consider the following (generalized) Cauchy's problem for Φ: Find the n-dimensional integral manifold of $\Sigma(\Phi)$ passing through a given non-characteristic one-dimensional integral manifold. Here by a manifold (in $J^m(M, N, \rho)$) we mean such a manifold that its projection to the space N does not degenerate in the usual sense. This Cauchy's problem possesses a unique solution by the Cartan-Kähler theorem (Cartan [1], [3], Kähler [5]). Let us try to solve Cauchy's problem by integrating a system of ordinary differential equations (as to the classical problem of this kind, in particular Darboux's method, refer to Goursat [4]). We consider this problem by distinguishing the following three cases. As before we write $\nu=s_1(X_0)$.

1°) *When $\nu=0$, Φ is completely integrable at X_0*: For each integral point X sufficiently near X_0, Φ possesses a unique solution passing through X. As is well-known, this solution can be obtained by integrating a system of ordinary differential equations.

2°) *When $\nu=1$, the solution of Cauchy's problem for Φ can be reduced to the integration of a system of ordinary differential equations.*

This fact follows from the following results due to E. Cartan:

(i) When $s_1=1$, $s_2=\cdots=s_n=0$, the differential system $\Sigma(\Phi)$ has $(n-1)$-dimensional Cauchy's characteristics (Cartan [1], p. 306).

(ii) The characteristic system of a differential system in the sense of Cartan is completely integrable (Cartan [3], Chap. III).

(iii) A completely integrable Pfaffian system can be solved by integrating a system of ordinary differential equations (Cartan [2], Chap. X; [3], Chap. III).

3°) *When $\nu>1$*, Cauchy's problem for Φ cannot be solved, in general, by integrating a system of ordinary differential equations.

However in a certain case we have a method of integration similar to Darboux's method in the case of two independent variables. Our method is roughly stated as follows. "*If* $(\nu-1)$ *different characteristic systems have respectively two independent invariants* (*including of higher order*) *each of which is independent of* Φ, *then the solution of Cauchy's problem can be reduced to the integration of a system of ordinary differential equations.*"

This follows from Theorem 4 and its generalization by combining the fact $2°$ above.

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