56. Note on Continuation of Real Analytic Solutions of Partial Differential Equations with Constant Coefficients*)

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In [1], [2] and [3] we have given some results on continuation of real analytic solutions of linear partial differential equations with constant coefficients to convex sets K of various types. In this note we remark that the assumption of the convexity of K can be much weakened. This problem has been presented by Professor S. Ito. Also I am indebted to Professor H. Komatsu for the improvement of the result. I am very greteful for their valuable suggestions.

Theorem 1. *Let K be a compact set in $\boldsymbol{R}^n$ such that $\boldsymbol{R}^n \setminus K$ is connected. Let $p(D)$ be a $t \times s$ matrix of linear partial differential operators with constant coefficients, and let p' be its transposed matrix. Assume that* $\mathrm{Hom}\,(\mathrm{Coker}\, p', \mathcal{P})=0$ *and that* $\mathrm{Ext}^1\,(\mathrm{Coker}\, p', \mathcal{P})$ *has no elliptic components, where $\mathcal{P}$ denotes the ring of polynomials. Then, for any open neighborhood U of K we have $A_p(U \setminus K)/A_p(U)=0$, namely, every real analytic solution of $p(D)u=0$ can be uniquely continued to U.*

Proof. Take $u \in A_p(U \setminus K)$. By the vanishing of the cohomology group $H^1(V, A)$ for any open set $V \subset \boldsymbol{R}^n$, we can take $f \in [A(\boldsymbol{R}^n \setminus K)]^s$ and $g \in [A(U)]^s$ such that

$$u=f-g \qquad \text{on } U \setminus K.$$

The assumption implies

$$0=p(D)u=p(D)f-p(D)g \qquad \text{on } U \setminus K.$$

Hence $p(D)f$ and $p(D)g$ define an element h of $A_{p_1}(\boldsymbol{R}^n)$, where p_1 is the compatibility system of p. Let $V \supset K$ be a relatively compact convex open set. Then by the existence theorem (see, e.g., [5], Theorem 1) we can find $V \in [A(V)]^s$ such that $p(D)v=h$ on V. Thus we have

$$f-v\,|_{V \setminus \mathrm{ch} K} \in A_p(V \setminus \mathrm{ch}\, K),$$

where $\mathrm{ch}\, K$ denotes the convex hull of K. By Theorem 2.3 of [2], we obtain a unique continuation $[f-v] \in A_p(V)$ of $f-v\,|_{V \setminus \mathrm{ch} K}$. Since $\boldsymbol{R}^n \setminus K$ is connected, $[f-v]$ agrees with $f-v$ whenever both are defined. Therefore

$$[u]=[f-v]+v-g$$

*) Partially supported by Fûjukai.

gives a real analytic continuation of u to the neighborhood $V\cap U$ of K. Considering the unique continuation property again, we have obtained the extension of u to $A_p(U)$. q.e.d.

We can give a more general result: The following is a real analytic version of the results of Komatsu [6], Theorem 4.1.

Theorem 2. *Let K be a compact subset of $\boldsymbol{R}^n$. Then for any open neighborhood U of K we have $H^1_K(U, A_p)\cong A_p(U\backslash K)/A_p(U)$. Hence the latter quotient space does not depend on U.*

Proof. We have the following long exact sequence in the general cohomology theory:

$$0\longrightarrow A_p(U)\longrightarrow A_p(U\backslash K)\longrightarrow H^1_K(U, A_p)$$
$$\longrightarrow H^1(U, A_p)\longrightarrow H^1(U\backslash K, A_p).$$

Thus it suffices to show that the restriction mapping $H^1(U, A_p)\to H^1(U\backslash K, A_p)$ is injective. Since the cohomology groups $H^k(V, A)$ vanish for $k\geqslant 1$ for any open subset $V\subset \boldsymbol{R}^n$, we can calculate $H^1(U, A_p)$ and $H^1(U\backslash K, A_p)$ employing the resolution

$$0\longrightarrow A_p\longrightarrow A^s\xrightarrow{p} A^t\xrightarrow{p_1} A^{t_2}\longrightarrow\cdots.$$

Thus we have

$$H^1(U, A_p)\cong A_{p_1}(U)/p(D)[A(U)]^s.$$
$$H^1(U\backslash K, A_p)\cong A_{p_1}(U\backslash K)/p(D)[A(U\backslash K)]^s.$$

Take a representative $u(x)\in A_{p_1}(U)$ of an element of $H^1(U, A_p)$ which goes to zero cohomology class by the restriction. This obviously implies that $u|_{U\backslash K}=p(D)v$ for some $v\in [A(U\backslash K)]^s$.

Now we consider v as a section of $\tilde{\mathcal{O}}^s$ on $U\backslash K$, where $\tilde{\mathcal{O}}$ denotes the sheaf of slowly increasing holomorphic functions on $\boldsymbol{D}^n\times i\boldsymbol{R}^n$; $\boldsymbol{D}^n$ is the directional compactification of $\boldsymbol{R}^n$ and $\tilde{\mathcal{O}}|_{\boldsymbol{R}^n}$ agrees with A (see [4]). We have $H^1(V, \tilde{\mathcal{O}})=0$ for any open set $V\subset \boldsymbol{D}^n$ ([4], Theorem 3.1.8). Thus we can find $f\in[\tilde{\mathcal{O}}(\boldsymbol{D}^n\backslash K)]^s$ and $g\in[A(U)]^s$ such that $v=f-g$ on $U\backslash K$. We have

$$p(D)f=p(D)v+p(D)g=u+p(D)g \qquad \text{on } U\backslash K.$$

Hence $p(D)f$ can be extended analytically to K. The extended element h obviously satisfies $p_1(D)h=0$, and belongs to $[\tilde{\mathcal{O}}(D)]^s$. The latter implies especially that h is holomorphic on a complex strip around $\boldsymbol{R}^n$ with a fixed breadth. Thus by the above quoted existence theorem ([5], Theorem 1) we can find $w\in[A(\boldsymbol{R}^n)]^s$ such that $p(D)w=h$. Thus we conclude that $u=p(D)(w-g)$ with $w-g\in[A(U)]^s$. This implies that u represents the zero cohomology class also in $H^1(U, A_p)$. The injectivity is proved. Due to the excision theorem $H^1_K(U, A_p)$ does not depend on U. q.e.d.

Finally we give a similar result for the situation in [3]. Since the sufficient conditions given there are complicated, we do not repeat them here.

Theorem 3. *Let K be the intersection of a compact set with the lower half space $\{x_n<0\}$. Assume that every irreducible component p_λ of a single linear partial differential operator $p(D)$ with constant coeffcients satisfies one of the following conditions*:

1) $\{x_n<0\}\backslash K$ *is connected and* p_λ *satisfies the condition of Theorem* 2.6 *in* [3].

2) K *and* p_λ *satisfy the condition of Theorem* 2.7, 2) *in* [3]. (*This time* $\{x_n<0\}\backslash K$ *is necessarily connected.*)

3) $\{x_n<0\}\backslash K$ *is connected and* p_λ *satisfies the condition of Theorem* 2.12 *in* [3].

Then we have $A_p(U\backslash K)/A_p(U)=0$.

The proof is similar. Though the application of the existence theorem diminishes the domain of analyticity to $\{x_n<-\delta\}$, we have no difficulty because δ is arbitrary and the solution is a fixed one.

References

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