109. On the Global Existence of Solutions of Differential Equations on Closed Subsets of a Banach Space

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1. Introduction. Let D be a subset of a real Banach space X and A be a continuous function from $[0, +\infty)\times D$ into X. In this paper we consider the initial value problem

(IVP) $$u'=A(t,u), \qquad u(0)=x,$$

where x is given in D. By a solution of (IVP) or of (IVP; x), we mean a continuously differentiable function u from $[0, +\infty)$ into D such that $u(0)=x$ and $u'(t)=A(t,u(t))$ for all $t\geqq 0$.

This kind of problem has been treated by many authors; for example, see Crandall [1], Lovelady-Martin [3], Martin [4], Pavel [5], [6] and the cited papers in them.

The purpose of this paper is to establish a global existence theorem for (IVP) under some conditions which are similar to those treated in [4] but somewhat weaker than them. Our theorem gives some simplifications and improvements of results in [4] and also provides an answer to a question raised by Martin [4].

2. Existence theorem. Let X be a real Banach space, X^* the dual space of X and denote by $\langle x, f\rangle$ the natural pairing between $x\in X$ and $f\in X^*$. For each $x, y\in X$, define

$$\langle y, x\rangle_i=\inf\{\langle y, f\rangle;\ f\in F(x)\},$$

where F is the duality mapping from X into X^*, i.e., F is defined by

$$F(x)=\{f\in X^*;\ \langle x, f\rangle=\|x\|^2=\|f\|^2\}$$

for each $x\in X$.

Now, let D be a closed subset of X, A a function from $[0, +\infty)\times D$ into X and consider the following conditions:

(A1) A is continuous from $[0, +\infty)\times D$ into X;

(A2) there is a real-valued continuous function ω defined on $[0, +\infty)$ such that

$$\langle A(t,x)-A(t,y), x-y\rangle_i\leqq\omega(t)\,\|x-y\|^2$$

for all (t,x) and (t,y) in $[0, +\infty)\times D$;

(A3) $\liminf_{h\to 0+} h^{-1}d(x+hA(t,x), D)=0$ for each (t,x) in $[0, +\infty)\times D$, where $d(z,D)$ stands for the distance from $z\in X$ to D.

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Then our result is the following:

Theorem. *Suppose that A satisfies the conditions* (A1), (A2) *and* (A3). *Then for each x in D there exists a unique global solution $u(\cdot\,; x)$ of* (IVP; x). *Furthermore, if x and y are in D, then*

$$(*) \qquad \|u(t; x)-u(t; y)\| \leqq \|x-y\| \exp\left(\int_0^t \omega(s)ds\right)$$

for all $t \geqq 0$.

Remark. This theorem gives an improvement of a result in Martin [4; Theorem 4]. In case $D=X$, this result is found in Lovelady-Martin [3] and Pavel [5], [6].

The inequality $(*)$, which ensures the uniqueness of solutions of (IVP), is obtained by the following lemma.

Lemma 1. *Let $x, y \in D$, $T>0$ and let u and v be solutions of* (IVP; y) *and of* (IVP; y) *on* $[0, T]$, *respectively. Then*

$$\|u(t)-v(t)\| \leqq \|x-y\| \exp\left(\int_0^t \omega(s)ds\right)$$

for all $t \in [0, T]$.

Proof. Putting $p(t)=\|u(t)-v(t)\|^2$, we have by (A2)

$$\begin{aligned} p'(t) &= 2\langle u'(t)-v'(t), u(t)-v(t)\rangle_i \\ &= 2\langle A(t, u(t))-A(t, v(t)), u(t)-v(t)\rangle_i \\ &\leqq 2\omega(t)p(t) \end{aligned}$$

for all $t \in (0, T)$. Hence we get the required inequality. Q.E.D.

Therefore to complete the proof of Theorem it suffices to show only the existence of solutions of (IVP). In the rest of this paper we shall do it.

3. Approximate solutions and local existence. In this section we establish the local existence of solutions of (IVP). In the following, we always assume the conditions (A1), (A2) and (A3).

We first note the following result which is proved in [8] (see also [7]).

Lemma 2. *Let $s \geqq 0$ be fixed. Then for each x in D there exists a unique continuous function u from* $[0, +\infty)$ *into D such that*

(i) $u(t)=x+\int_0^t A(s, u(r))dr$ *for all $t \geqq 0$;*

(ii) $\|A(s, u(t))\| \leqq \exp(\omega(s)t)\,\|A(s, x)\|$ *for all $t \geqq 0$.*

Using Lemma 2, we can construct approximate solutions of (IVP) in the following sense.

Proposition 1. *Let $x \in D$ and $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \varepsilon_n=0$. Then there are positive numbers M_x and T_x such that for each $n \geqq 1$, there exists a partition $\{0=t_0^n<t_1^n<\cdots<t_{N_n}^n=T_x\}$ of $[0, T_x]$ and a continuous function u_n from $[0, T_x]$ into D satisfying the properties:*

(i) $t_{k+1}^n-t_k^n \leqq \varepsilon_n$ *for all k, $k=0, 1, \cdots, N_n-1$;*

(ii) *$u_n(0)=x$ and $\|u_n(t)-u_n(s)\| \leqq M_x|t-s|$ for all $t, s \in [0, T_x]$;*

(iii) *u_n is continuously differentiable in each interval (t_k^n, t_{k+1}^n) and $u_n'(t)=A(t_k^n, u_n(t))$ for $t \in (t_k^n, t_{k+1}^n)$;*

(iv) *if $t \in [t_k^n, t_{k+1}^n]$, then $\|A(t, u_n(t))-A(t_k^n, u_n(t))\| \leqq \varepsilon_n$.*

Proof. We use a technique similar to Martin [4]. Fix a positive number T_0 sufficiently large. Then by (A1), there are positive numbers R and M such that

$$\|A(t, y)\| \leqq M \quad \text{for all } t \in [0, T_0] \quad \text{and} \quad y \in D \cap B_R(x),$$

where $B_R(x)=\{y \in X; \|y-x\| \leqq R\}$. Put $M_x=M \sup\{\exp(T_0|\omega(t)|); t \in [0, T_0]\}$ and choose a positive number T_x so that $T_x \leqq \min\{T_0, R/M_x\}$.

Now, for each $n \geqq 1$, let $t_0^n=0$ and v_0^n be a continuous function from $[0, +\infty)$ into D such that

$$v_0^n(t)=x+\int_0^t A(0, v_0^n(r))dr \qquad \text{for all } t \geqq 0.$$

Inductively define a number t_{k+1}^n in $[0, T_x]$ and a continuous function v_{k+1}^n from $[0, +\infty)$ into D in the following manner: Choose a number h_k^n in $[0, \varepsilon_n]$ such that

(1) $t_k^n+h_k^n \leqq T_x$;

(2) if $t \in [t_k^n, t_k^n+h_k^n]$, then $\|A(t, v_k^n(t-t_k^n))-A(t_k^n, v_k^n(t-t_k^n))\| \leqq \varepsilon_n$;

(3) h_k^n is the largest number in $[0, \varepsilon_n]$ such that (1) and (2) hold.

Then define $t_{k+1}^n=t_k^n+h_k^n$ and v_{k+1}^n to be a unique continuous function from $[0, +\infty)$ into D such that

$$v_{k+1}^n(t)=v_k^n(h_k^n)+\int_0^t A(t_{k+1}^n, v_{k+1}^n(r))dr \qquad \text{for all } t \geqq 0.$$

Note that by (A1), $h_k^n>0$ if $t_k^n<T_x$ and also that by Lemma 2, such a function v_{k+1}^n always exists. We now assert that there is a positive integer N_n such that $t_{N_n-1}^n<T_x$ and $t_{N_n}^n=T_x$. Assume, for contradiction, that $t_k^n<T_x$ for all k and $\lim_{k\to+\infty} t_k^n=c \leqq T_x$. Then it follows from Lemma 2 that

$$\begin{aligned}\|v_0^n(h_0^n)-x\| &\leqq h_0^n \exp(|\omega(0)|\, h_0^n)\, \|A(0, x)\| \\ &\leqq M_x h_0^n \\ &\leqq R\end{aligned}$$

and

$$\|v_{k+1}^n(h_{k+1}^n)-v_k^n(h_k^n)\| \leqq h_{k+1}^n \exp(|\omega(t_{k+1}^n)|\, h_{k+1}^n)\, \|A(t_{k+1}^n, v_k^n(h_k^n))\|$$

for all $k \geqq 0$. These imply that $v_k^n(h_k^n) \in D \cap B_R(x)$ for all $k \geqq 0$ and

$$\|v_{k+1}^n(h_{k+1}^n)-v_k^n(h_k^n)\| \leqq M_x h_{k+1}^n \qquad \text{for all } k \geqq 0.$$

Therefore, $z=\lim_{k\to+\infty} v_k^n(h_k^n)$ exists in D. Using (A1), take a $\delta>0$ so that $\delta \leqq \varepsilon_n$ and $\|A(t, y)-A(c, z)\| \leqq \varepsilon_n/2$ whenever $|t-c| \leqq \delta$ and $\|y-z\| \leqq 2\delta M_x$. Let k_0 be sufficiently large so that $c-t_k^n \leqq \delta$ and $\|z-v_k^n(h_k^n)\| \leqq \delta M_x$ for all $k \geqq k_0$. If $k \geqq k_0+1$, then

$$\begin{aligned}\|v_k^n(t-t_k^n)-z\| &\leqq \|v_k^n(t-t_k^n)-v_{k-1}^n(h_{k-1}^n)\|+\|v_{k-1}^n(h_{k-1}^n)-z\| \\ &\leqq (t-t_k^n)\exp\{|\omega(t_k^n)|\,(t-t_k^n)\}\, \|A(t_k^n, v_{k-1}^n(h_{k-1}^n))\|+\delta M_x\end{aligned}$$

$$\leqq (t-t_k^n)M_x + \delta M_x$$
$$\leqq 2\delta M_x$$

for all $t \in [t_k^n, c]$ and hence

$$\|A(t, v_k^n(t-t_k^n)) - A(t_k^n, v_k^n(t-t_k^n))\|$$
$$\leqq \|A(t, v_k^n(t-t_k^n)) - A(c, z)\| + \|A(c, z) - A(t_k^n, v_k^n(t-t_k^n))\|$$
$$\leqq \varepsilon_n$$

for all $t \in [t_k^n, c]$. Therefore, by the definition of h_k^n, it must be true that $h_k^n \geqq c - t_k^n$, i.e., $t_{k+1}^n = t_k^n + h_k^n \geqq c$, which contradicts the fact that $t_{k+1}^n < c$. Thus we have shown that there is a positive integer N_n such that $t_{N_n-1}^n < T_x$ and $t_{N_n}^n = T_x$. Now, define u_n on $[0, T_x]$ by

$$u_n(t) = v_k^n(t-t_k^n) \quad \text{if } t \in [t_k^n, t_{k+1}^n], \quad k = 0, 1, \cdots, N_n - 1.$$

Then by the definitions of t_k^n and v_k^n, it is easy to see that $\{t_k^n\}_{0 \leqq k \leqq N_n}$ and u_n have the required properties. Q.E.D.

We now apply Proposition 1 to prove the following:

Proposition 2. *For each x in D, there is a positive number T_x such that* (IVP; x) *has a unique solution on* $[0, T_x]$.

Proof. Let $x \in D$ and $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\lim_{n \to +\infty} \varepsilon_n = 0$. Let M_x and T_x be the same positive numbers as in Proposition 1. Also, let u_n be the continuous functions obtained in Proposition 1. For simplicity, put $L = \sup\{|\omega(t)|; t \in [0, T_x]\}$ and $p(t) = \|u_n(t) - u_m(t)\|^2$ for each $t \in [0, T_x]$. If $t \in (t_k^n, t_{k+1}^n) \cap (t_j^m, t_{j+1}^m)$, then we have by Proposition 1 and (A2),

$$\begin{aligned} p'(t) &= 2\langle u_n'(t) - u_m'(t), u_n(t) - u_m(t)\rangle_i \\ &= 2\langle A(t_k^n, u_n(t)) - A(t_j^m, u_m(t)), u_n(t) - u_m(t)\rangle_i \\ &\leqq 2\langle A(t, u_n(t)) - A(t, u_m(t)), u_n(t) - u_m(t)\rangle_i \\ &\quad + 2(\varepsilon_n + \varepsilon_m)\|u_n(t) - u_m(t)\| \\ &\leqq 2\omega(t)p(t) + 4(M_x T_x + \|x\|)(\varepsilon_n + \varepsilon_m) \\ &\leqq 2Lp(t) + 4(M_x T_x + \|x\|)(\varepsilon_n + \varepsilon_m). \end{aligned}$$

Since this differential inequality holds true for all $t \in [0, T_x]$ except a finite number of points in $[0, T_x]$, it follows that

$$\|u_n(t) - u_m(t)\|^2 \leqq 4(M_x T_x + \|x\|)(\varepsilon_n + \varepsilon_m) \int_0^t e^{2L(t-s)} ds$$

for all $t \in [0, T_x]$. Thus the sequence $\{u_n\}$ is uniformly Cauchy on $[0, T_x]$. For each $t \in [0, T_x]$, define $u(t) = \lim_{n \to +\infty} u_n(t)$. Then it is clear that $u(0) = x$, $u(t) \in D$ for all $t \in [0, T_x]$ and u is Lipschitz continuous on $[0, T_x]$. Now, since $\|u_n'(t) - A(t, u_n(t))\| \leqq \varepsilon_n$ whenever $t \in (t_k^n, t_{k+1}^n)$, we see that

$$\left\| u_n(t) - \left(x + \int_0^t A(s, u_n(s)) ds\right) \right\| \leqq T_x \varepsilon_n \qquad \text{for all } t \in [0, T_x].$$

Also, by (A1), we see that $\lim_{n \to +\infty} A(t, u_n(t)) = A(t, u(t))$ uniformly for $t \in [0, T_x]$. Therefore, passing to the limit, we obtain that

$$u(t)=x+\int_0^t A(s,u(s))ds \qquad \text{for all } t\in[0,T_x].$$

Thus u is a unique solution of (IVP; x) on $[0,T_x]$. Q.E.D.

4. Global existence. Before proceeding to the proof of Theorem, we prepare the following lemma.

Lemma 3. *Let $x\in D$ and let T be a positive number such that (IVP; x) has a solution on $[0,T]$. Then there is a positive number R such that for every $y\in D\cap B_R(x)$, (IVP; y) has a solution on $[0,T]$.*

Proof. Let $x\in D$ and u be a unique solution of (IVP; x) on $[0,T]$. Since the set $\{(t,u(t));t\in[0,T]\}$ is compact in $[0,T]\times D$, (A1) ensures that there are positive numbers r and M such that

$$\|A(t,y)\|\leqq M \qquad \text{for all } t\in[0,T] \text{ and all } y\in D\cap B_r(u(t)).$$

Choose a positive number R so that

$$R\leqq r\exp\left(-\int_0^T|\omega(s)|\,ds\right).$$

Now, let $y\in D\cap B_R(x)$ and T_y be a positive number such that (IVP; y) has a solution v on $[0,T_y)$. Assume that v is non-continuable and $T_y\leqq T$. Then, by Lemma 1,

$$\|u(t)-v(t)\|\leqq\|x-y\|\exp\left(\int_0^t\omega(s)ds\right)\leqq r \qquad \text{for all } t\in[0,T_y),$$

i.e., $v(t)\in D\cap B_r(u(t))$ for all $t\in[0,T_y)$. Hence

$$\|A(t,v(t))\|\leqq M \qquad \text{for all } t\in[0,T_y),$$

so that

$$\|v(t)-v(s)\|\leqq M\,|t-s| \qquad \text{for all } t,s\in[0,T_y).$$

This implies that $\lim_{t\to T_y}v(t)$ exists in D, which contradicts the fact that v is non-continuable. Thus $T<T_y$ and the proof is completed.

Q.E.D.

Proof of Theorem. Let $x\in D$ and let C be a connected component of D containing x. Let T be a positive number such that (IVP; x) has a solution on $[0,T]$. Now, consider the set

$$E=\{y\in C;\ (\text{IVP};\ y) \text{ has a solution on } [0,T]\}.$$

Then $E\neq\phi$, because $x\in E$. By Lemma 3, E is relatively open in C. Also, E is relatively closed in C. In fact, let $\{y_n\}$ be any sequence in E which converges to $y\in C$ in the topology of C. Then it is clear that $\lim_{n\to+\infty}\|y_n-y\|=0$. Therefore, it follows from Lemma 1 that the solution v_n of (IVP; y_n) on $[0,T]$ converges to a continuous function v from $[0,T]$ into D uniformly on $[0,T]$ as $n\to+\infty$. Clearly this limit v is a solution of (IVP; y) on $[0,T]$. Hence $y\in E$. Thus it must hold that $E=C$. Since C is a connected component of D, this fact implies that (IVP; x) has a solution on $[0,kT]$ for any integer $k\geqq 1$ and hence it is proved that (IVP; x) has a solution on $[0,+\infty)$. Q.E.D.

Remark. The idea for the proof of global existence is also found in [2].

Addeed in proof. The authors are most grateful to Professor R. H. Martin, Jr. for pointing out a gap in the proof of the theorem. According to his advices, the final part of the proof of the theorem should be as follows: Let b be any positive number. Then our proof of Proposition 1 shows that there exists $\delta>0$ such that (IVP) with initial time s and initial data x has a solution on $[s, s+\delta]$ for any $s \in [0, b]$. Taking δ as T and noting that our argument in the proof of the theorem remains valid when we take any $s>0$ instead of 0, we see that (IVP; x) has a solution on $[0, b]$.

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