48. On Symmetric Structure of a Group

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1. Introduction. Let A be a set and S a mapping of A into the symmetric group on A. Denote the image of $a(\in A)$ under S by S_a or $S[a]$ and the image of $x(\in A)$ under S_a by xS_a. Then S is called a *symmetric structure* of A if the following conditions are satisfied:

(i) $aS_a=a$, (ii) $S_a^2=I$ (the identity), (iii) $S[bS_a]=S_aS_bS_a$. A set with a symmetric structure is called a *symmetric set.* A symmetric set A is called *effective* if $a\neq b$ implies $S_a\neq S_b$. Then group generated by $\{S_aS_b \mid a, b\in A\}$ is called the *group of displacements* and is denoted by $G(A)$. A symmetric structure of a finite set has been studied in [1] and [2].

Now let A be a group. Then A has symmetric structure S defined by $xS_a=ax^{-1}a$. The purpose of this note is to study the structure of $G(A)$ for a given group A, and we shall determine it when the center $Z(A)$ of A is trivial.

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2. Group of displacements. In this section we assume that A is a group and S is a symmetric structure of A defined above.

Proposition 1. *A is effective if and only if there is no involution in the center of A.*

Proof. Let $Z(A)$ be the center of A, and assume that $Z(A)$ contains an involution t. Then $xS_{at}=(at)x^{-1}(at)=ax^{-1}a=xS_a$. Therefore A is not effective.

Conversely, assume that A is not effective, then there exist distinct two elements a and b in A such that $S_a=S_b$. Therefore, for any element x in A,

(1) $$ax^{-1}a=bx^{-1}b.$$

Replacing x with e (the unit element) and a, we have

(2) $$a^2=b^2$$

(3) $$a=ba^{-1}b.$$

Then $b^{-1}a=(ab^{-1})^{-1}$ by (2), $(ab^{-1})^2=e$ by (3) and $(b^{-1}a)x^{-1}(ab^{-1})=x^{-1}$ for any x in A. Hence, $ab^{-1}\in Z(A)$ and $(ab^{-1})^2=e$. Thus $Z(A)$ contains an involution.

Let L_a and R_a be permutations on A such that

$$L_a\colon x\rightarrow ax,$$

$$R_a: x \rightarrow xa.$$

Then $\mathfrak{L}=\{L_a \mid a \in A\}$ and $\mathfrak{R}=\{R_a \mid a \in A\}$ are permutation groups on A. $\mathfrak{L}$ is anti-isomorphic to A, $\mathfrak{R}$ is isomorphic to A and $\mathfrak{L}$ and $\mathfrak{R}$ commute elementwise. If $Z(A)=\{e\}$, then the permutation group $\langle \mathfrak{L}, \mathfrak{R} \rangle = \mathfrak{L}\mathfrak{R}$ on A which is generated by $\mathfrak{L}$ and $\mathfrak{R}$, is isomorphic to the direct product of $\mathfrak{L}$ and $\mathfrak{R}$.

Proposition 2. *$G(A)$ is generated by $\{L_a R_a \mid a \in A\}$.*

Proof. Since $x(S_a S_b) = ba^{-1}xa^{-1}b = x(L_a^{-1}R_a^{-1})(L_b R_b)$, $G(A) \subseteq \langle L_a R_a \mid a \in A \rangle$. Conversely, $x(L_a R_a) = axa = x(S_e S_a)$, and hence $\langle L_a R_a \mid a \in A \rangle \subseteq G(A)$.

Corollary. *If $Z(A)=\{e\}$, then $G \subseteq \mathfrak{L} \times \mathfrak{R}$.*

Let H be the full set of an element h which satisfies the following:

(*) There exist some elements $a_1, a_2, \cdots, a_r$ in A such that $h = a_1 a_2 \cdots a_r$ and $a_r a_{r-1} \cdots a_1 = e$.

Proposition 3. *If $Z(A)=\{e\}$, then we have the following:*

(i) *$H = A'$ (the commutator subgroup of A).*

(ii) *$G(A) \cap \mathfrak{L} = \{L_h \mid h \in H\}$.*

(iii) *$G(A) \cap \mathfrak{R} = \{R_h \mid h \in H\}$.*

Proof. $Z(A)=\{e\}$ implies that A is effective and $\langle \mathfrak{L}, \mathfrak{R} \rangle = \mathfrak{L} \times \mathfrak{R}$.

(i) It is easily seen that H is a normal subgroup of A. For any elements a and b in A, $[a, b] = a^{-1}b^{-1}(ab)$ and $(ab)b^{-1}a^{-1} = e$. Hence $A' \subseteq H$. Conversely, let $\bar{a}$ be a coset of A' in A which contains a, then for any h in H

$$\bar{h} = \bar{a}_1 \bar{a}_2 \cdots \bar{a}_r = \bar{a}_r \bar{a}_{r-1} \cdots \bar{a}_1 = \bar{e}.$$

It follows that $H \subseteq A'$.

(ii) If $P \in G(A) \cap \mathfrak{L}$, then there exists $b_1, b_2, \cdots, b_s$ in A such that

$$P = (L_{b_1} R_{b_1})(L_{b_2} R_{b_2}) \cdots (L_{b_s} R_{b_s}).$$

Hence $(L_{b_s b_{s-1} \cdots b_1})(R_{b_1 b_2 \cdots b_s})$ is in $\mathfrak{L}$. It follows that $R_{b_1 b_2 \cdots b_s}$ is the identity permutation on A. Therefore, we have

$$P = L_{b_s b_{s-1} \cdots b_1} \quad \text{and} \quad b_1 b_2 \cdots b_s = e.$$

By the same argument in (ii), we have (iii).

Proposition 4. *If $Z(A)=\{e\}$, then*

$$G(A) = \{L_h L_a R_{h'} R_a \mid a \in A, h, h' \in H\}.$$

Proof. By (ii) and (iii) of Proposition 3, we have

$$\{L_h L_a R_{h'} R_a \mid a \in A, h, h' \in H\} \subseteq G(A).$$

Conversely, for any element P in $G(A)$, there exist some elements $b_1, b_2, \cdots, b_s$ in A such that

$$P = (L_{b_s b_{s-1} \cdots b_1})(R_{b_1 b_2 \cdots b_s}).$$

By (i) of Proposition 3, there exists some element h in H such that $b_s b_{s-1} \cdot \cdots \cdot b_1 = b_1 b_2 \cdots b_s h$, hence

$$P = L_h (L_{b_1 b_2 \cdots b_s})(R_{b_1 b_2 \cdots b_s}).$$

It follows that $G(A) \subseteq \{L_h L_a R_{h'} R_a \mid a \in A, h, h' \in H\}$.

Theorem 1. *If $Z(A)=\{e\}$ and let N be a $G(A)$-orbit in A which contains e, then N is a normal subgroup of A and A/N is an elementary abelian group of exponent* 2.

Proof. By Proposition 4, x is contained in N if and only if $x=aha$ for some a in A and h in H. Therefore, by (i) of Proposition 3,

$$N=\{aha \mid a \in A, h \in H\}=\bigcup_{a\in A} aHa$$
$$=\bigcup_{a\in A} aA'a=\bigcup_{a\in A} a^2A'.$$

It follows that N is a normal subgroup of A and for any element a in A, a^2 is contained in N.

We denote $G(A)\cap\mathfrak{L}$ and $G(A)\cap\mathfrak{R}$ by $\bar{\mathfrak{L}}$ and $\bar{\mathfrak{R}}$ respectively.

Theorem 2. *If $Z(A)=\{e\}$, then*

$$G(A)/\bar{\mathfrak{L}}\times\bar{\mathfrak{R}}=A/A'.$$

Proof. By Proposition 4, for any element P in A' such that $P=L_hL_aR_{h'}R_a$. Let ϕ be a mapping of $G(A)$ into A/A' such that

$$\phi: P=L_hL_aR_{h'}R_a\rightarrow aA'.$$

Then it is easily seen that ϕ induces an isomorphism of $G(A)/\bar{\mathfrak{L}}\times\bar{\mathfrak{R}}$ onto A/A'.

From Proposition 3 and Theorem 2, we have the following,

Corollary. *If $Z(A)=\{e\}$ and $A=A'$, then $G(A)=\mathfrak{L}\times\mathfrak{R}$.*

References

[1] N. Nobusawa: On symmetric structure of a finite set. Osaka J. Math., **11**, 569–575 (1974).

[2] M. Kano, H. Nagao, and N. Nobusawa: On finite homogeneous symmetric sets (to appear).