44. *On a Conjecture of Regge and Sato on Feynman Integrals*

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The purpose of this note is to show that any (unrenormalized) Feynman integral should satisfy a holonomic system of linear differential equations. This result gives an affirmative answer to the conjecture given in an illuminating report by Regge [7], who first understood and emphasized the importance of the role of the differential equations in the investigation of Feynman integrals. This important property of the Feynman integral has also been conjectured and proved in simple cases by Sato [8] independently and in a little different context. See also Barucchi-Ponzano [1], Kawai-Stapp [5] and references cited there. Note that Kawai-Stapp [5] discusses the S-matrix itself, not the individual Feynman integrals, as Sato [8] proposes. The characteristic variety of the holonomic system discussed here enjoys a nice physical interpretation, as is shown by Kashiwara-Kawai-Stapp [4]. In this note we discuss the generalized Feynman integral after Speer [10]. For simplicity we assume that all relevalent particles are spinless. However, we do not necessarily assume that their masses are different from zero.

Renormalized integrals will be discussed in our subsequent papers.

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Our arguments rely on following lemmas. (Cf. Kashiwara-Kawai [3], Kashiwara [2] and Kashiwara-Kawai-Stapp [4].)

Lemma 1. *Let $\varphi_j(x)$ $(j=1,\cdots,d)$ and $f_l(x)$ $(l=1,\cdots,N)$ be a real valued real analytic functions defined on a real analytic manifold M. Denote by X a complexification of M. Denote by Y the variety defined by $\{x\in X;\ \varphi_1(x)=\cdots=\varphi_d(x)=0\}$. Assume that Y has codimension d in X and that Y is irreducible and non-singular except for proper analytic subset Y_{sing} of Y. Assume that $f_l|_Y\not\equiv 0$ $(l=1,\cdots,N)$ and that*

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Y_{sing} *is contained in* $\left\{x\in X;\ \prod_{l=1}^{N} f_l(x)=0\right\}$. *Assume that complex numbers* λ_l $(l=1,\cdots,N)$ *satisfy the condition that* $\sum_{l=1}^{N}\lambda_l\nu_l$ *never becomes a rational number for any integer* ν_l $(l=1,\cdots,N)$. *Then* $\Phi(x)=\prod_{j=1}^{d}\delta(\varphi_j(x))\prod_{l=1}^{N}(f_l(x)+i0)^{\lambda_l}$ *is a well-defined hyperfunction which satisfies a holonomic system* $\mathcal{M}$ *of linear differential equations. Further the characteristic variety of* $\mathcal{M}$ *is contained in* $W_0'=\Big\{(x,\eta)\in T^*X$; *there exist a sequence* $x_m\in X$ *such that* $\varphi_j(x_m)=0$ $(j=1,\cdots,d)$ *and that converges to* x *with* $\prod_{l=1}^{N} f_l(x)=0$ *and a sequence* $s_m=(s_1^{(m)},\cdots,s_N^{(m)})\in \boldsymbol{C}^N$ *and* $t_m=(t_1^{(m)},\cdots,t_d^{(m)})\in\boldsymbol{C}^d$ *such that* $s_l^{(m)}f_l(x_m)$ *converges to zero* $(l=1,\cdots,N)$ *and that* $\sum_{l=1}^{N}s_l^{(m)}\operatorname{grad}_x f_l(x_m)+\sum_{j=1}^{d}t_j^{(m)}\operatorname{grad}_x\varphi_j(x_m)$ *converges to* $\eta\Big\}$.

Lemma 2. *Assume the same conditions as in Lemma* 1. *Assume further* $Y_{\mathrm{sing}}=\emptyset$. *Then the singularity spectrum of* $\Phi(x)$ *is confined to the following set* $W_0'(+)$.

$W_0'(+)=\Big\{(x,\sqrt{-1}\eta)\in\sqrt{-1}S^*M$; *there exist a sequence* $x_m\in X$ *such that* $\varphi_j(x_m)=0$ $(j=1,\cdots,d)$ *and that converges to* x *with* $\prod_{l=1}^{N}f_l(x)=0$ *and a sequence* $s_m=(s_1^{(m)},\cdots,s_N^{(m)})\in(\boldsymbol{R}^+)^N$ *and* $t_m=(t_1^{(m)},\cdots,t_d^{(m)})\in\boldsymbol{C}^d$ *such that* $s_l^{(m)}f_l(x_m)$ *converges to z ero and* $\sum_{l=1}^{N}s_l^{(m)}\operatorname{grad}_x f_l(x_m)+\sum_{j=1}^{d}t_j^{(m)}\operatorname{grad}_x\varphi_j(x_m)$ *converges to* $\eta\Big\}$.

The generalized Feynman integral $F_D(p,\lambda)$ associated with a Feynman diagram D with n vertices, n' external lines and N internal lines is by definition the following integral (1) up to a constant factor. (Cf. Nakanishi [6], Speer [10] and the references cited there.)

$$(1)\qquad \int \frac{\prod_{j=1}^{n}\delta^4\left(\sum_{r=1}^{n'}[j:r]p_r+\sum_{l=1}^{N}[j:l]k_l\right)}{\prod_{l=1}^{N}(k_l^2-m_l^2+i0)^{\lambda_l}}\prod_{l=1}^{N}d^4k_l.$$

Since this integral is not a proper integral, we consider the corresponding integral defined on a projective compactification $P(\boldsymbol{R}^{4N})$ of $\boldsymbol{R}^{4N}$, i.e. discuss the integral by extending the integrand to $\boldsymbol{R}^{4n'}\times P(\boldsymbol{R}^{4N})$ in a natural way so that Lemma 1 and Lemma 2 can be applied to the extended integrand. Such an extension is possible, as long as we are dealing with generalized Feynman integrals with generic λ. The integral thus defined on the projective compactification will also be denoted by $F_D(p,\lambda)$.

Then, by Lemma 1 combined with (7) of Kashiwara-Kawai [3] and Lemma 2 combined with Theorem 2.3.1 in Chapter I of Sato-Kawai-Kashiwara [9] we obtain the following:

Theorem. *Generalized Feynman integral $F_D(p,\lambda)$ with generic λ satisfies a holonomic system $\mathcal{M}$ of linear differential equations whose characteristic variety is contained by the following set $\mathcal{L}$ (extended Landau variety), where X and M denoted $\boldsymbol{C}^{4n'}$ and $\boldsymbol{R}^{4n'}$, respectively.*

$\mathcal{L}=\{(p,u)\in T^*X$; there exist a sequence of scalars $c^{(m)}$ and $\alpha_l^{(m)}$ $(l=1,\cdots,N)$ and four-vectors $p_r^{(m)}$ $(r=1,\cdots,n')$, $u_r^{(m)}$ $(r=1,\cdots,n')$, $k_l^{(m)}$ $(l=1,\cdots,N)$ and $v_j^{(m)}$ $(j=1,\cdots,n)$ which satisfy the following relations (2)$\}$.

$$(2)\qquad \begin{cases} p_r^{(m)}\to p_r & (r=1,\cdots,n') \\ u_r^{(m)}\to u_r & (r=1,\cdots,n') \\ \sum_{r=1}^{n'}[j:r]p_r^{(m)}+\sum_{l=1}^{N}[j:l]k_l^{(m)}=0 & (j=1,\cdots,n) \\ u_r^{(m)}=\sum_{j=1}^{n}[j:r]v_j^{(m)} & (r=1,\cdots,n') \\ \dfrac{\sum_{j=1}^{n}[j:l]v_j^{(m)}+\alpha_l^{(m)}k_l^{(m)}}{c^{(m)}}\to 0 & (l=1,\cdots,N) \\ \alpha_l^{(m)}(k_l^{(m)2}-m_l^2)\to 0 & (l=1,\cdots,N) \\ c^{(m)} \quad \text{is bounded} & \\ c^{(m)}k_l^{(m)} \quad \text{is bounded for } l=1,\cdots,N & \\ (c^{(m)},c^{(m)}k_1^{(m)},\cdots,c^{(m)}k_N^{(m)})\not\to 0. & \end{cases}$$

Further, the singularity spectrum of $F_D(p,\lambda)$ with generic λ is confined to the set $\mathcal{L}^+\subset\sqrt{-1}S^*M$, where $\mathcal{L}^+$ is defined in the same way as $\mathcal{L}$ with the additional condition that the limiting point (p,u) should belong to S^*M and that $c^{(m)}$ is a real number, $(c^{(m)},c^{(m)}k_1^{(m)},\cdots,c^{(m)}k_N^{(m)})$ converges to a real point and $\alpha_l^{(m)}$ is a non-negative real number $(l=1,\cdots,N)$.

The details of this note will be given somewhere else.

References

[1] Barucchi, G., and G. Ponzano: J. Math. Phys., **14**, 396 (1973).
[2] Kashiwara, M.: On the rationality of roots of b-functions (to appear).
[3] Kashiwara, M., and T. Kawai: Proc. Japan Acad., **51**, 270 (1975).
[4] Kashiwara, M., T. Kawai, and H. P. Stapp: Micro-analytic structure of the S-matrix and related functions (to appear).
[5] Kawai, T., and H. P. Stapp: Lecture Notes in Physics, No. 39, Springer, 38 (1975).
[6] Nakanishi, N.: Graph Theory and Feynman Integrals. Gordon and Breach (1971).
[7] Regge, T.: Report of Battelle Rencontres. Benjamin, 433 (1968).
[8] Sato, M.: Lecture Notes in Physics, No. 39, Springer, 13 (1975).

[9] Sato, M., Kawai, T., and M. Kashiwara: Lecture Notes in Math. No. 287. Springer, 265 (1973).
[10] Speer, E. R.: Generalized Feynman Amplitudes. Princeton Univ. Press (1969).