94. *Prime Closed Geodesics on Pinched Spheres**

By Sadaharu YOKOYAMA
Suzuka College of Technology, Suzuka Mie Japan

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In this article we generalize the notion of type number for an abstract variation problem and count the number of prime closed geodesics on a pinched sphere (Theorem 3.2, 3.3).

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1. Definition 1.1. Let $(X, \varphi; f)$ be a triple of a topological space X, a continuous function φ on X such that $\varphi \geq 0$ and a continuous function φ on X such that $\varphi \geq 0$ and a continuous map f into itself. The triple $(X, \varphi; f)$ is a (abstract) variation problem (over a field k) if X, φ, and f satisfies the following:

i) $\varphi(f(x)) \leq \varphi(x)$ for any $x \in X$.

ii) $\varphi(f(x)) = \varphi(x)$ implies $f(x) = x$.

iii) the homomorphism f_* induced by f on $H_*(X; k)$ is the identity.

A point $x \in X$ for which $f(x) = x$ is said to be a critical point of $(X, \varphi; f)$ and the totality of the critical point is denoted by γ_f.

Definition 1.2. Let $(X, \varphi; f)$ be a variation problem. A norm $|A|$ of A is defined by the following, for any compact set A in X

$$|A| = \sup \left\{ \lim_{n \to \infty} \varphi(f^n(x)) : x \in A \right\}.$$

Then the triple $(X, \varphi; f)$ is said to have a norm if the norm above satisfies the following:

iv) for any compact set A and for any neighborhood U of $\gamma_f \cap \varphi^{-1}(|A|)$, there is an integer N such that $n \geq N$ implies $f^n(A) \subset U \cup \varphi^{-1}([0, |A|))$.

Definition 1.3. A variation problem $(X, \varphi; f)$ is said to be discrete if

i) the set $(\gamma_f \cap \varphi^{-1}(a))'$ is discrete for any real number $a \geq 0$, where $(*)'$ is the derived set of $(*)$.

ii) $\varphi(\gamma_f)$ is discrete in real number.

Definition 1.4. Let X be a topological space and φ be a continuous function on X. Then an n-th type number $T_n(x; X, \varphi)$ of x is defined by the following:

$$T_n(x; X, \varphi) = \varprojlim H_n(U, U^-; k)$$

*) Dedicated to Professor Ryoji Shizuma on his 60th birthday.

where $\{U\}$ is a neighborhood system of x and $U^{-}=\{y \in U : \varphi(y) < \varphi(x)\}$.

Definition 1.5. A homology class $\alpha \in H_*(X ; k)$ is said to be subordinate to $\beta \in H_*(X ; k)$ and is denoted by $\alpha \prec \beta$, if there is a cohomology class $\xi \in H^m(X ; k)$ $(m \geq 1)$ such that $\alpha = \beta \cap \xi$.

Theorem in [6] is improved as follows.

Theorem 1.6. *Let $(X, \varphi ; f)$ be a discrete variation problem with a norm over a locally connected space X, and if there is a sequence $\{\alpha_i\}_{i=1}^{k}$ of homology class of X such that*

i) $\alpha_1 \notin \operatorname{Ker} j_* : H_*(X ; k) \rightarrow H_*(X, X^a ; k)$

ii) $\alpha_1 \prec \alpha_2, \prec, \cdots, \prec \alpha_k$

then there exist critical points $x_1, x_2, \cdots, x_k$ such that

iii) $a < \varphi(x_1) < \varphi(x_2) < \cdots < \varphi(x_k)$

iv) $T_{n(i)}(x ; X, \varphi) \neq 0 \qquad n(i) = \dim \alpha_i$

where j is an inclusion map and $X^a = \{x \in X : \varphi(x) < a\}$.

Corollary 1.7. *Let (X, Y, F, π) be a fiber bundle over a locally connected base space Y and a compact metric bundle space X, and $(X, \varphi ; f)$ be a discrete variation problem with a norm such that the restriction of $\pi^{-1}(y)$ is constant for any $y \in Y$. If there is a sequence $\{\alpha_i\}_{i=1}^{k}$ of homology class of Y such that*

i) $\alpha_1 \notin \operatorname{Ker} j_* : H_*(Y ; k) \rightarrow H_*(Y, Y^a ; k)$

ii) $\alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_k$

then there exist critical points $x_1, x_2, \cdots, x_k \in X$ snch that

iii) $a < \varphi(x_1) < \varphi(x_2) < \cdots < \varphi(x_k)$.

iv) $T_{n(i)}(\pi(x_i) ; Y, \varphi \cdot \pi^{-1}) \neq 0$

where j is an inclusion map and $Y^a = \{\pi(x) : x \in X^a\}$.

2. Denote by M a compact smooth riemannian manifold without boundary and denote by A, B submanifolds of M.

Definition 2.1.

1) $\Omega(M ; A, B)$ consists of all piecewise-smooth curves α in M which is parametrized by the real number t $(0 \leq t \leq 1)$, proportionate to arc length and $\alpha(0) \in A$, $\alpha(1) \in B$.

2) a metric d on $\Omega(M ; A, B)$ is defined by

$$d(\alpha, \beta) = \sup \{\rho(\alpha(t), \beta(t)) : 0 \leqslant t \leqslant 1\} + \left| \int_0^1 |\dot{\alpha}(t)| - |\dot{\beta}(t)| \right| dt$$

where ρ is riemannian metric on M.

3) Finally a continuous function on $\Omega(M ; A, B)$ is defined by

$$L : \alpha \rightarrow \int_0^1 |\dot{\alpha}(t)|\, dt.$$

Theorem 2.2. *Together with the canonical Morse deformation f_t on $\Omega^a(M ; A, B)$, the triple $(\Omega^a(M ; A, B), L ; f_1)$ turns out to be a normed variation problem such that*

i) *for any $(x, y) \in A \times B$ the restriction of f_t on $\Omega^a(M ; x, y)$ defines a deformation on $\Omega^a(M ; x, y)$, where $\Omega(M ; x, y) = \Omega(M ; \{x\}, \{y\})$.*

ii) *for any closed set* F *in* $\Omega^a(M;A,B)$ *satisfying* $F\cap\gamma_f=\emptyset$ *there is a real number* $\delta>0$, *such that*

$$\text{“if } \beta\in F, \text{ then } L(\beta)-L(f_1(\beta))>\delta\text{”.}$$

iii) *all critical points are geodesics.*

Proof. The proof is similar to that in [5].

Proposition 2.3. *Let* α *be a geodesic of the variation problem* $(\Omega^a(M;A,B),L;f_1)$ *of 2.2. If* $\alpha(0)$ *and* $\alpha(1)$ *are not conjugate along* α, *then there exist a neighborhood* U *of* α *in* $\Omega^a(M;A,B)$ *and a neighborhood* V *of* $(\alpha(0),\alpha(1))$ *in* $A\times B$ *satisfying the following:*

$*$) *there is a unique geodesic* $n(x,y)$ *in* U *such that* $n(x,y)(0)=x$, $n(x,y)(1)=y$ *for any* $(x,y)\in V$, *depending continuously on* $(x,y)\in V$.

Let $(\Omega^a(M;A,B),L;f_1)$ be the variation problem and I' be a continuous function on $A\times B$, then define a continuous function I on $\Omega(M;A,B)$ by the following:

$$I:\beta\mapsto I'(\beta(0),\beta(1)),$$

then $(\Omega^a(M;A,B),L+I;f_1)$ become a variation problem. Let α be a geodesic of the variation problem $(\Omega^a(M;A,B),L;f_1)$, and assume $\alpha(0)$ and $\alpha(1)$ are not conjugate along α, a function $\tilde{I}$ on the neighborhood V in $A\times B$ of 2.3 given by

$$\tilde{I}:(x,y)\mapsto L(n(x,y))+I'(x,y)$$

is continuous.

Theorem 2.4. *Let* α *be a geodesic of the variation problem* $(\Omega^a(M;A,B),L+I;f_1)$, *and* I' *be a continuous function on* $A\times B$. *Assume* $\alpha(0)$ *and* $\alpha(1)$ *are not conjugate along* α, *then*

$$T_*(\alpha;\Omega^a(M;A,B),L+I)\simeq T_*((\alpha(0),\alpha(1));V,\tilde{I})$$
$$\otimes T_*(\alpha;\Omega^a(M;\alpha(0),\alpha(1));L)$$

Outline of proof. For $(x,y)\in V$, $\beta\in U$ the join by geodesics of x and y to $\beta(0)$ and $\beta(1)$, respectively, gives a continuous map $g((x,y),\beta)$ of $U\times V$ into $\Omega(M;A,B)$, and set

$$U_0=\{\beta\in U:(L+I)(\beta)<(L+I)(\alpha)\}$$
$$V_0=\{(x,y)\in V:\tilde{I}(x,y)<\tilde{I}(\alpha(0),\alpha(1))\}$$
$$W=\{U\cap\Omega(M;\alpha(0),\alpha(1))\}$$
$$W_0=\{\beta\in W:L(\beta)<L(\alpha)\}$$
$$U_1=\{\beta\in U_0:(\beta(0),\beta(1))\in V_0 \text{ or } g((\alpha(0),\alpha(1)),\beta)\in W_0\}$$
$$Y_1=\{((x,y),\beta)\in V\times W):(x,y)\in V_0 \text{ or } g((x,y),\beta)\in U_0\}.$$

We compute the homology of the space (U,U_0) in terms of the homology of the space $(V,V_0),(W,W_0)$. Starting with spaces U_1,Y_1, we reduce the computation of homology $\varprojlim H_*(U,U_0)$ and $\varprojlim H_*(V,V_0)\otimes\varprojlim H_*(W,W_0)$ into $\varprojlim H_*(U,U_1)$ and $\varprojlim H_*(V\times W,Y_1)$, respectively, by successive approximation.

3. Let M be a compact smooth $m+1$ dimensional riemannian manifold without boundary, and c be a closed geodesic on M and c^n also be a closed geodesic that satisfies $c^n(t)=c(nt)$ where $c(0)=c(1)$.

Let $\Delta=\{(x,x)\in M\times M\}$, and let elements in $\Omega(M,\Delta)$ be parametrized by $S^1=[0,1]/\{0,1\}$. Then $0(2)$ act on $\Omega(M,\Delta)$, and set $\Pi(M)=\Omega(M\Delta)/0(2)$.

Set

$$T_*(c)=T_*(c;\Pi(M),L) \qquad T_*(c,)=T_*(c;\Omega(M;c(0),c(1)),L)$$
$$T(c)=\inf\{s: T_s(c)\neq 0\} \qquad T(c,)=\inf\{s: T_s(c,)\neq 0\}$$
$$\bar{T}(c)=\sup\{s: T_s(c)\neq 0\} \qquad \bar{T}(c,)=\sup\{s: T_s(c,)\neq 0\}.$$

Theorem 3.1.

i) $\bar{T}(c^n)-T(c^n,)\leq 2m$,

ii) $T(c^{sn})\geq(s-1)\bar{T}(c^n,)$,

iii) $(\bar{T}(c^s)-2m)/s\leq\lim_{n\to\infty}T(c^n)/n\leq\lim_{n\to\infty}\bar{T}(c^n)/n\leq(T(c^s)+2m)/s$.

Proof. This theorem is proved easily using 2.4.

The following theorems can be deduced from the knowledge of the homology group of $\Pi(M)$ using 1.7, 3.1.

Theorem 3.2. *Let M^m be a riemannian manifold which is homeomorphic to S^m $(m>2)$. If the sectional curvature K of M satisfies $\frac{1}{4}<K\leq 1$, then there is a subset of $\Pi(M)$ satisfying following;*

i) *F is the set of the closed geodesics of the same length.*

ii) *the derived set F' of F is not discrete.*

Theorem 3.3. *Let M^{2k} be a riemannian manifold, homeomorphic to S^{2k} and let K satisfy $0<l<K\leq 1$. If the number of prime closed geodesics on M is greater than p, then the following inequality holds:*

$(2\sqrt{l}(2k-1)/4k-t))(]g(2k)-t)/p[)\leq 1$ *t: nonnegative integer, where $]*[$ denotes the smallest integer exceeds $*$ and $g(m)=2m-s(m)-1$ with $s(m)=m-2^h$, $0\leq s(m)\leq 2^h$.*

References

[1] R. Bott: On the iteration of closed geodesics and the Sturm intersection theory. Comm. Pure Appl. Math., **9**, 171–206 (1956).

[2] W. Klingenberg: Manifolds with restricted conjugate locus. Ann. of Math., **78**, 527–547 (1963).

[3] J. Milnor: Morse Theory. Ann. of Math., Studies, No. 51, Princeton University Press, Princeton (1963).

[4] M. M. Postnikov: The variation theory of geodesics. W. B. Saunders. Co. (1967).

[5] H. Seifert and W. Threlfall: Variations rechnung im Grossen. Chelsea Publ. Co. (1948).

[6] Y. Shikata and I. Mogi: Some topological aspects of abstract variation theory. Differential Geometry in honor of K. Yano, 451–457 (1972).

[7] R. Shizuma and Y. Shikata: A note on variation theory. Proc. Japan Acad., **45**, 517–521 (1969).