136. *The Concrete Description of the Colocalization*

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(Communicated by Kenjiro SHODA, M. J. A., Nov. 12, 1976)

Introduction. Recently K. Ohtake [5] proved that for a torsion theory $(\mathcal{T}, \mathcal{F})$ there is a colocalization functor if and only if $\mathcal{F}$ is a TTF-class, in this case we have another torsion theory $(\mathcal{F}, \mathcal{D})$ and T. Kato [2], K. Ohtake [5] proved that there is an equivalence between the colocalization subcategory $[C]$ of Mod-R with respect to $(\mathcal{T}, \mathcal{F})$ and the localization subcategory $[L]$ of Mod-R with respect to $(\mathcal{F}, \mathcal{D})$.

In this paper, we shall show a colocalization of any module M_R can be obtained by $M \otimes_R I \otimes_R I$ concretely where I is a corresponding two sided ideal, i.e. the unique minimal ideal belonging to the filter which corresponds to $(\mathcal{F}, \mathcal{D})$.

As an application of this, we get self-contained and fairly simple proofs of the results in [5].

The concrete description of the colocalization. Throughout this paper, ring R means an associative ring with unit, Mod-R (resp. R-Mod) denotes a class of all unital right (resp. left) R-modules and $(\mathcal{A}, \mathcal{B})$ (resp. $(\mathcal{A}^*, \mathcal{B}^*)$) denotes a torsion theory in Mod-R (resp. R-Mod), about which the reader is referred to [6].

Let $(\mathcal{A}, \mathcal{B})$ be a torsion theory. A module M_R is called "*divisible*" if $Ext^1_R(K, M)=0$ for any $K \in \mathcal{A}$, dually "*codivisible*" if $Ext^1_R(M, K)=0$ for any $K \in \mathcal{B}$, and a map $M_R \xrightarrow{f} L(M)_R$ (resp. $C(M)_R \xrightarrow{f} M_R$) is called "*localization*" of M_R (resp. "*colocalization*" of M_R) if $\ker(f)$, $\mathrm{cok}(f) \in \mathcal{A}$, $L(M)_R \in \mathcal{B}$ and $L(M)$ is divisible (resp. $\ker(f) \in \mathcal{B}$, $\mathrm{cok}(f) \in \mathcal{B}$, $C(M) \in \mathcal{A}$ and $C(M)$ is codivisible).

$[L]$, $[C]$ denote the full subcategory of torsion-free divisible modules in Mod-R and torsion codivisible modules in Mod-R which are called localization subcategory and colocalization subcategory with respect to $(\mathcal{A}, \mathcal{B})$ respectively.

Let I be a two sided idempotent ideal and $\mathcal{F}=\{M_R \in \text{Mod-}R \mid M \cdot I=0\}$, then $\mathcal{F}$ is TTF-class in the sense of Jans [1]. (i.e. closed under taking submodules, extensions and direct products). Any TTF-class in Mod-R is obtained as above, in this case corresponding torsion class and torsion-free class are $\mathcal{T}=\{M_R \mid M \cdot I=M\}$ and $\mathcal{D}=\{M_R \mid \mathrm{Ann}_M(I)=0\}$ respectively. (i.e. $(\mathcal{T}, \mathcal{F})$, $(\mathcal{F}, \mathcal{D})$ are torsion theories.) The corresponding filter with respect to $(\mathcal{F}, \mathcal{D})$ is $\mathcal{J}=\{J_R \mid J_R$ is a right ideal which contains $I\}$ and torsion submodule of M_R with respect to $(\mathcal{T}, \mathcal{F})$ is $M \cdot I$. (see [1]). Any TTF-theory in R-Mod denoted by $(\mathcal{T}^*, \mathcal{F}^*)$, $(\mathcal{F}^*, \mathcal{D}^*)$ is defined by the same way. These notations are maintained throughout this paper.

Localization functor with respect to $(\mathcal{F}, \mathcal{D})$ is represented by $L(-) = \mathrm{Hom}_R(I_R, \mathrm{Hom}_R(I_R, -)_R) = \mathrm{Hom}_R(I \otimes_R I_R, -)$. (See [6], by a simple calculation of operator, new operation of R to $\mathrm{Hom}_R(I_R, -)$ coincides with the origin.) On the other hand, for the formulation of the colocalization, we get the following theorem.

Theorem 1. *Colocalization functor with respect to* $(\mathcal{T}, \mathcal{F})$ *is obtained by* $C(-) = - \otimes_R I \otimes_R I$ *(*$C(M_R) \xrightarrow{f} M$ *is canonical). Particularly* $C(R) = I \otimes_R I$.

Proof. We show that $M \otimes_R I \otimes_R I \xrightarrow{f} M$ where $f(\sum m \otimes i_1 \otimes i_2) = \sum m i_1 i_2 \in M$, $i_1, i_2 \in I$ is a colocalization of M. In the proof we will omit the suffix to avoid the unnecessary confusion. Let $\sum m \otimes i_1 \otimes i_2 \in \ker(f)$, $i_1, i_2 \in I$ and $m \in M$, i.e. $\sum m i_1 i_2 = 0$. We can write $i = \sum j_1 j_2$, $j_1, j_2 \in I$ for any $i \in I$ since $I^2 = I$, so $(\sum m \otimes i_1 \otimes i_2) i = \sum m \otimes i_1 \otimes (i_2 j_1 j_2) = \sum (\sum m i_1 i_2) \otimes j_1 \otimes j_2 = 0$. Hence $\ker(f) I = 0$ so $\ker(f) \in \mathcal{F}$. It is clear $\mathrm{cok}(f) = M/MI \in \mathcal{F}$, $M \otimes_R I \otimes_R I \in \mathcal{T}$.

We shall show $M \otimes_R I \otimes_R I$ is codivisible. Let $0 \longrightarrow A_R \longrightarrow B_R \xrightarrow{t} C_R \longrightarrow 0$ be an exact sequence in Mod-R such that $A_R \in \mathcal{F}$, then we have exact sequences and natural isomorphisms

$$\begin{array}{ccc} 0 = \mathrm{Hom}(M \otimes_R I \otimes_R I_R, A_R) & \longrightarrow & \mathrm{Hom}(M \otimes_R I \otimes_R I_R, B_R) \\ \wr\wr & & \wr\wr \\ 0 = \mathrm{Hom}(M \otimes I, \mathrm{Hom}(I, A)) & \longrightarrow & \mathrm{Hom}(M \otimes I, \mathrm{Hom}(I, B)) \\ & \longrightarrow & \mathrm{Hom}(M \otimes_R I \otimes_R I_R, C_R) \\ & & \wr\wr \\ & \xrightarrow{h} & \mathrm{Hom}(M \otimes I, \mathrm{Hom}(I, C)) \end{array}$$

since $I \in \mathcal{T}$, $A \in \mathcal{F}$ where $h = \mathrm{Hom}(M \otimes_R I, \mathrm{Hom}(I, t))$.

We must verify h is an epimorphism. Let $g: M \otimes_R I_R \to \mathrm{Hom}_R(I, C)_R$ be any R-homomorphism, $M \otimes_R I \in \mathcal{T}$ hence $\mathrm{Im}(g) \in \mathcal{T}$ so $\mathrm{Im}(g) \subset \mathrm{Hom}_R(I, C)I$. Hence to show this we prove $p = \mathrm{Hom}(I, t) | \mathrm{Hom}(I, B)I$ is an isomorphism onto $\mathrm{Hom}(I, C)I$. Clearly $\mathrm{Hom}(I, t)$ is monomorphism, so is p. Let $\sum yi \in \mathrm{Hom}(I, C)I$ where $y \in \mathrm{Hom}(I, C)$, $i \in I$. We can write $i = \sum i_1 i_2$, $i_1, i_2 \in I$ for $I^2 = I$. Consider the map $y i_1: I_R \to C_R$ and $y i_1(j) = y(i_1 j) = y(i_1) j$ for $j \in I$. Since $y(i_1) \in C$ and t is an epimorphism, there is $b_{i_1} \in B$ such that $t(b_{i_1}) = y(i_1)$. So we define $k_{i_1}: I_R \to B_R$ such that $k_{i_1}(j) = b_{i_1} \cdot j$ for any $j \in I$, then $t \cdot k_{i_1} i_2(j) = t \cdot k_{i_1}(i_2 \cdot j) = t(b_{i_1} \cdot i_2 j) = t(b_{i_1}) i_2 j = y(i_1) i_2 j = y \cdot i_1 i_2(j)$ for any $j \in I$, hence $t \cdot k_{i_1} i_2 = y \cdot i_1 i_2$ so we put $q_i = \sum k_{i_1} \cdot i_2$ then $q_i \in \mathrm{Hom}(I, B)I$ and $t q_i = \sum t k_{i_1} i_2 = \sum y i_1 i_2 = y(\sum i_1 i_2) = yi$, that is $\sum yi = t(\sum q_i)$, $\sum q_i \in \mathrm{Hom}(I, B)I$, which means p is an epimorphism, so is h, as was to be shown.

Localization and colocalization functors are unique up to the isomorphism if they exist (see [3], [4], [6]), therefore identity map $C(M) \to C(M)$ is a colocalization of $C(M)$, hence $C(C(M)) \cong C(M)$ so we get the following lemma.

Lemma 2. *$M \otimes_R I \otimes_R I \otimes_R I \otimes_R I \xrightarrow{f} M \otimes_R I \otimes_R I$ is an isomorphism where I is any two sided idempotent ideal and $f(\sum m \otimes i_1 \otimes i_2 \otimes i_3 \otimes i_4) = \sum m \otimes i_1 \otimes i_2 \cdot (i_3 i_4)$. Particularly ${}_R(I \otimes_R I) \otimes_R (I \otimes_R I)_R \cong {}_R I \otimes_R I_R$.*

Using above results, we get the next theorem immediately. (2)–(5) have been proved in [5].

Theorem 3. *Let I be a two sided idempotent ideal, $(\mathcal{T}, \mathcal{F})(\mathcal{F}, \mathcal{D})$; $(\mathcal{T}^*, \mathcal{F}^*)(\mathcal{F}^*, \mathcal{D}^*)$ corresponding torsion theories, C, C^*; L, L^* localization and colocalization functors with respect to $(\mathcal{T}, \mathcal{F})(\mathcal{T}^*, \mathcal{F}^*)$; $(\mathcal{F}, \mathcal{D})$ $(\mathcal{F}^*, \mathcal{D}^*)$ respectively and $[C], [L]$ colocalization and localization subcategories respectively.*

The following statements hold.

(1) ${}_R C(R)_R \cong {}_R C^*(R)_R$ *as $R-R$ bi-homomorphism.*

(2) *Bilinear map $(I \otimes_R I, I \otimes_R I) \xrightarrow{v} I \otimes_R I$ where $v(\sum i_1 \otimes i_2, \sum i_3 \otimes i_4) = (\sum i_1 \otimes i_2) \cdot (\sum i_3 i_4)$ gives a ring structure in $C(R), C^*(R)$ and colocalization $C(R) \to R$, (1) are ring and $R-R$ bi-homomorphism.*

(3) *$C(R)^2 = C(R)$ and if R is commutative, so is $C(R)$.*

(4) *Functors C, L induce an equivalence $[C] \sim [L]$.*

(5) *$[C]$ is a Grothendieck category.*

Proof. (1)–(3) are obvious by Theorem 1.

Proof of (4): For any $M_R \in$ Mod-R, $C(M) \in [C]$ and $L(M) \in [L]$, it remains to show $C(L(M)) \cong M$ for $M \in [C]$ and $L(C(M)) \cong M$ for $M \in [L]$. But by Lemma 2 and uniqueness of the localization, it is sufficient to show that $\mathrm{Hom}_R(I \otimes_R I, M_R) \otimes_R I \otimes_R I_R \cong M \otimes_R I \otimes_R I_R$ and $\mathrm{Hom}_R(I \otimes_R I_R, M \otimes_R I \otimes_R I_R)_R \cong \mathrm{Hom}_R(I \otimes_R I_R, M_R)_R$ canonically.

Let $M_R \in$ Mod-R. The latter is induced by the colocalization $M \otimes_R I \otimes_R I_R \xrightarrow{f} M_R$. Since $\ker(f) \in \mathcal{F}$, $\mathrm{cok}(f) \in \mathcal{F}$ and $I \otimes_R I$ is torsion codivisible, we have exact sequences

$$0 = \mathrm{Hom}_R(I \otimes_R I, \ker(f)) \to \mathrm{Hom}_R(I \otimes_R I, M \otimes_R I \otimes_R I) \to \mathrm{Hom}_R(I \otimes_R I, \mathrm{Im}(f)) \to 0$$

$$0 \to \mathrm{Hom}_R(I \otimes_R I, \mathrm{Im}(f)) \to \mathrm{Hom}_R(I \otimes_R I, M) \to \mathrm{Hom}_R(I \otimes_R I, \mathrm{cok}(f)) = 0.$$

Hence $\mathrm{Hom}(I \otimes_R I, f) : L(C(M)) \to L(M)$ is isomorphism.

Next we must verify the former is induced by the localization $M_R \xrightarrow{g} \mathrm{Hom}_R(I \otimes_R I, M)_R$. Since $\ker(g) \in \mathcal{F}$, $\mathrm{cok}(g) \in \mathcal{F}$, we get a commutative diagram for $M_R \in [C]$:

$$\begin{array}{ccccc} 0\longrightarrow \ker(g\otimes I\otimes I) & \longrightarrow & M\otimes_R I\otimes_R I & \xrightarrow{g\otimes I\otimes I} & \mathrm{Hom}_R\,(I\otimes_R I, M)\otimes_R I\otimes_R I \\ \downarrow g_1 & & \wr\wr\downarrow g_2 & & \downarrow g_3 \\ 0\longrightarrow \ker(g) & \longrightarrow & M & \xrightarrow{g} & \mathrm{Hom}_R\,(I\otimes_R I, M) \end{array}$$

$$\longrightarrow(\mathrm{cok}\,(g)\otimes_R I\otimes I)=0$$

$$\longrightarrow\mathrm{cok}\,(g)\longrightarrow 0$$

where rows are exact, g_2 and g_3 are canonical map and g_1 is an induced map. g_1 is monomorphism and $\ker(g)\in\mathcal{F}$ so is $\ker(g\otimes I\otimes I)$. However $\mathrm{Hom}_R\,(I\otimes_R I, M)\otimes_R I\otimes_R I$ is codivisible with respect to $(\mathcal{T},\mathcal{F})$, hence above row splits, so $\ker(g\otimes I\otimes I)=0$ for $M\otimes_R I\otimes_R I\in\mathcal{T}$. Combining with two isomorphisms $\mathrm{Hom}\,(I\otimes_R I, f)$ and $(h\otimes I\otimes I)g_2^{-1}$ where h is a localization $M\otimes_R I\otimes_R I_R\to\mathrm{Hom}_R\,(I\otimes_R I, M\otimes_R I\otimes_R I)$, for any $M_R\in$ Mod-R, we have natural isomorphisms:

$$\begin{aligned} M\otimes_R I\otimes_R I &\cong \mathrm{Hom}_R\,(I\otimes_R I, M\otimes_R I\otimes_R I)\otimes_R I\otimes_R I \\ &\cong \mathrm{Hom}_R\,(I\otimes_R I, M)\otimes_R I\otimes_R I \end{aligned}$$

whose composition is $g\otimes I\otimes I$ by a routine calculation. Hence $g\otimes I\otimes I: C(M)\to C(L(M))$ is isomorphism.

Proof of (5): Clear from (4).

References

[1] J. P. Jans: Some aspect of torsions. Pacific J. Math., **15**, 1249–1259 (1965).
[2] T. Kato: Duality between Colocalization and Localization (to appear).
[3] J. Lambek: Lectures on Rings and Modules. Blaisdell (1966).
[4] R. J. McMaster: Cotorsion theories and colocalization. Can. J. Math., **27**(3), 618–628 (1975).
[5] K. Ohtake: Colocalization and Localization (to appear).
[6] Bo Stenström: Rings and Modules of Quotient. Springer Lecture Notes, **237** (1971).