149. On a Pair of Groups and its Sylow Bases

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Only finite groups are to be considered in this note. Any unexplained notation and terminology should be referred to [1] and [2]. Throughout this note, let A and B be groups such that a pair $(A, B: f, g)$ of groups is well defined, where $f: A \to B$ and $g: B \to A$ are homomorphisms and let $|A|=|B|=p_1^{e_1}\cdots p_n^{e_n}$, where the p's are different primes and each e_i is a positive integer. Suppose A is solvable. Then B is also solvable. In this case, we shall call $(A, B: f, g)$ *solvable*. By P. Hall ([3]), the classical theorems about Sylow subgroups have been extended to the Sylow systems of a solvable group. With respect to $(A, B: f, g)$ which is solvable, we will give the following which are analogous to P. Hall's results. We denote by $\{S_i\}_n$ $(\{T_i\}_n)$ a set of Sylow p_i-subgroups $S_i(T_i)$ of $A(B)$, $i=1, \cdots, n$, respectively.

Theorem 1. *Let $(A, B: f, g)$ be solvable and $\{S_i\}_n$ a Sylow basis of A. Then there is a Sylow basis $\{T_i\}_n$ of B such that for each $i=1, \cdots, n$, $(S_i, T_i: f, g)$ is well defined.*

The set $\{(S_i, T_i: f, g)\}_n$ given in Theorem 1 is called a *Sylow basis* of $(A, B: f, g)$.

Theorem 2. *Let $(A, B: f, g)$ be solvable, let $(M, N: f, g)$ be a subgroup of $(A, B: f, g)$ and $\{(P_i, Q_i: f, g)\}_m$ with $m \leqslant n$ a Sylow basis of $(M, N: f, g)$, where each P_i has order a power of p_i. Then there is a Sylow basis $\{(S_i, T_i: f, g)\}_n$ of $(A, B: f, g)$ such that for each $i=1, \cdots, m$, $(M \cap S_i, N \cap T_i: f, g)$ is well defined and equal to $(P_i, Q_i: f, g)$.*

Corollary. *Let $(A, B: f, g)$ be solvable and let $\{(S_i, T_i: f, g)\}_m$ with $m \leqslant n$ be a set of Sylow p_i-subgroups $(S_i, T_i: f, g)$ of $(A, B: f, g)$, $i=1, \cdots, m$, such that for each $i, j=1, \cdots, m$, $S_iS_j=S_jS_i$ and $T_iT_j=T_jT_i$. Then there is a Sylow basis $\{(S_i, T_i: f, g)\}_n$ of $(A, B: f, g)$ which contains $\{(S_i, T_i: f, g)\}_m$.*

To prove those theorems, we prepare some lemmas. Let π denote a set of primes and $(M, N: f, g)$ a subgroup of $(A, B: f, g)$ such that M is a π-subgroup (a Hall π-subgroup) of A. Then N is also a π-subgroup (a Hall π-subgroup) of B. In this case, we shall call $(M, N: f, g)$ a *π-subgroup* (a *Hall π-subgroup*) of $(A, B: f, g)$. The following is well known.

Lemma 1. *Let H be a Hall π-subgroup of a solvable group A and $M \lhd A$. Then $H \cap M$ and MH/M are Hall π-subgroups of M and A/M,* *respectively.*

Lemma 2. *Let $(A, B: f, g)$ be solvable and $\{(S_i, T_i: f, g)\}_n$ a Sylow basis of $(A, B: f, g)$. Then, for any subset $\{i_1, \cdots, i_r\}$ of $\{1, \cdots, n\}$, $(S_{i_1}\cdots S_{i_r}, T_{i_1}\cdots T_{i_r}: f, g)$ is well defined.*

Proof. By Lemma 1, $f(A)\cap T_{i_1}\cdots T_{i_r}$ and $f(S_{i_1}\cdots S_{i_r})$ are Hall $\{p_{i_1}, \cdots, p_{i_r}\}$-subgroups of $f(A)$. Furthermore $f(S_{i_1}\cdots S_{i_r})\subseteq f(A)\cap T_{i_1}\cdots T_{i_r}$. Hence $f(S_{i_1}\cdots S_{i_r})=f(A)\cap T_{i_1}\cdots T_{i_r}$. Similarly $g(T_{i_1}\cdots T_{i_r})=g(B)\cap S_{i_1}\cdots S_{i_r}$. Hence our result follows from [2, Lemma 1].

Lemma 3. *Let $(A, B: f, g)$ be solvable, let H be a Hall π-subgroup of A and K a subgroup of B. Then $(H, K: f, g)$ is well defined iff K is a Hall π-subgroup of $g^{-1}(H)$ and $f(H)\subseteq K$. In this case, $(H, K: f, g)$ is a Hall π-subgroup of $(A, B: f, g)$.*

Proof. Let $M=g(B)\cap H$. Then M is a Hall π-subgroup of $g(B)$. Let T be a Hall π-subgroup of B. Since $g(T)$ is a Hall π-subgroup of $g(B)$, there is $b\in B$ such that $g(b)^{-1}g(T)g(b)=M$. Thus $b^{-1}Tb\subseteq g^{-1}(M)$. Hence any Hall π-subgroup K of $g^{-1}(H)$ is a Hall π-subgroup of B. From this fact and Lemma 1, it follows that $f(H)=f(A)\cap K$ and $g(K)=g(B)\cap H$ if $f(H)\subseteq K$. Hence the "if" part holds. The "only if" part holds clearly.

Using Lemma 1 and Lemma 3, we obtain the following lemma and remark by the same way as in proofs of [2, Theorem 2] and [2, Theorem 3], respectively.

Lemma 4. *Let $(A, B: f, g)$ be solvable and $(P, Q: f, g)$ a π-subgroup of $(A, B: f, g)$. Then there is a Hall π-subgroup $(H, K: f, g)$ of $(A, B: f, g)$ such that $(P, Q: f, g)$ is a subgroup of $(H, K: f, g)$.*

Remark. Let $(A, B: f, g)$ be solvable, let $(M, N: f, g)$ be a normal subgroup of $(A, B: f, g)$ and $(H, K: f, g)$ a Hall π-subgroup of $(A, B: f, g)$. Then $(M\cap H, N\cap K: f, g)$, $(MH/M, NK/N: \bar{f}, \bar{g})$ and $(MH, NK: f, g)$ are well defined where $\bar{f}$ and $\bar{g}$ are homomorphisms which are naturally induced by f and g, respectively.

Proof of Theorem 1. For each $i=1, \cdots, n$, set $\pi_i=\{p_j \mid j\neq i\}$ and $H_i=\langle S_j \mid j\neq i\rangle$. Then each H_i is a Hall π_i-subgroup of A. By Lemma 3, there is a Hall π_i-subgroup K_i of B such that $(H_i, K_i: f, g)$ is well defined. Set $T_i=\bigcap_{j\neq i} K_j$. Then $\{T_i\}_n$ is a Sylow basis of B (cf. Proof of [1, Theorem 4.3.5]). Furthermore $S_i=\bigcap_{j\neq i} H_j$ and so

$$f(S_i)\subseteq\bigcap_{j\neq i} f(H_j)\subseteq\bigcap_{j\neq i} K_j=T_i\subseteq\bigcap_{j\neq i} g^{-1}(H_j)=g^{-1}(S_i).$$

Hence, by Lemma 3, $(S_i, T_i: f, g)$ is well defined.

Proof of Theorem 2. Set $\Sigma_1=\{1, \cdots, m\}$ and $\Sigma_2=\{m+1, \cdots, n\}$. Furthermore set $H_i=\langle P_j \mid j\neq i,\ j\in\Sigma_1\rangle$, $K_i=\langle Q_j \mid j\neq i,\ j\in\Sigma_1\rangle$, $\pi_i=\{p_j \mid j\neq i,\ j\in\Sigma_1\}$ and $\Pi_i=\{p_j \mid j\neq i,\ j\in\Sigma_1\cup\Sigma_2\}$. Then, by Lemma 2, $(H_i, K_i: f, g)$ is well defined and a Hall π_i-subgroup of $(M, N: f, g)$. By Lemma 4, there is a Hall Π_i-subgroup $(H_i^*, K_i^*: f, g)$ of $(A, B: f, g)$ such that for $i \in \Sigma_1$, it contains $(H_i, K_i : f, g)$ and for $i \in \Sigma_2$, it contains $(M, N : f, g)$. Set $S_i = \bigcap_{j \neq i} H_j^*$ and $T_i = \bigcap_{j \neq i} K_j^*$. Then, by the same way as in the proof of Theorem 1, we have that $(S_i, T_i : f, g)$ is well defined. Furthermore $\{S_i\}_n$ and $\{T_i\}_n$ are Sylow bases of A and B, respectively. Hence $\{(S_i, T_i : f, g)\}_n$ is a Sylow basis of $(A, B : f, g)$. Since $S_i \cap M = P_i$ and $T_i \cap N = Q_i$ for $i \in \Sigma_1$, this completes our proof.

Proof of Corollary. Let $H = S_1 \cdots S_m$ and $K = T_1 \cdots T_m$. Then $(H, K : f, g)$ is well defined and a subgroup of $(A, B : f, g)$. Furthermore $\{(S_i, T_i : f, g)\}_m$ is a Sylow basis of $(H, K : f, g)$. Now our assertion follows at once from Theorem 2.

References

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