33. *On an Extension of Pólya's "Ganzwertige ganze Funktion".*

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Mr. Pólya treated the integral functions $g(z)$ which take integral values for all integral values of z and called them "ganzwertige ganze Funktionen"[1]. I have tried to extend this idea in the following way:

1. Let us consider a set of positive integers

$$Z: \quad (z_1, z_2, \cdots\cdots\cdots)$$

and a function $g(z)$, which takes integral values (in the rational corpus or imaginary quadratic corpus) for all z_i. Denote by $\pi(n)$ the number of z_i's which is not greater than n and put $\operatorname{Max}_{|z|\leqq r} g(z) = M(r)$. We construct a function $\Psi(x)$, which coincides with $\pi(x)$ for all integral values of x and is otherwise linear in x. With this $\Psi(x)$ we form also a function $\varphi(x)$, which is continuously differentiable and such that $\varphi(0) = 0$ and $\varphi(x) \leqq \Psi(x)$ for $x > 0$. Then we have:

Theorem A. *If we can chose a real function $\rho = \rho(r)$ such that*

$$(r+1)\log r - r - \int_0^r \varphi'(x)\log(\rho - x)dx + \log \rho M(\rho) \longrightarrow -\infty$$

and $$\int_0^r \frac{\varphi(x)}{1+x}dx - \log M(r) \longrightarrow +\infty \qquad \textit{as} \quad r \longrightarrow +\infty$$

then $g(z)$ must be a polynomial.

We can prove this theorem by means of a method, similar to Pólya's, save as we have to evaluate a quantity of the form

$$\prod_{i=1}^{n}(z - z_i) \quad \text{for} \quad |z| = r.$$

2. As the special cases of this theorem we have:

If one of the following conditions is satisfied:

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1) Pólya, Über die ganzwertige ganze Funktionen, Rend. Palermo, 40 (1915), 1–16. For the literature see my paper under the same title, Tohoku mathematical Journal 27 (1926), 41–52.

a) $\pi(n) \geqq n-(k-c)\log n$ and $M(r) \leqq \dfrac{2^r}{r^{k\log r}}$,

b) $\pi(n) \geqq n-O(n^k)$ and $M(r) \leqq (2-c)^r$,

c) $\pi(n) \geqq n-\dfrac{kn}{\log(n+e)}$ and $M(r) \leqq (1+e^{-k}-c)^r$,

d) $\pi(\pi) \geqq kn$ and $M(r) \leqq (e-c)^{r^k}$,

e) $\pi(n) \geqq \dfrac{kn}{\log n}$ and $M(r) \leqq \exp\{(e-c)^{\sqrt{k\log r}}\}$,

f) $\pi(n) \geqq n^k$ and $M(r) \leqq \exp\left[\exp\left\{\left(\dfrac{1}{1-k}+c\right)\mathrm{loglog}\, r\right\}\right]$,

then $g(z)$ must be a polynomial.

3. From d) it follows that

if
$$\overline{\lim}\,\frac{\mathrm{loglog}\,M(r)}{\log r}=k,$$
then
$$\overline{\lim}\,\frac{\pi(n)}{n}\leqq k.$$

Mr. Skolem has called $\lim\limits_{n\to\infty}\dfrac{\pi(n)}{n}$ the *durchschnittliche Dichte* of $\pi(n)$[1]. This result means that the durchschnittliche Dichte of the lattice point on the curve $y=g(x)$ is not greater than
$$\overline{\lim}\,\frac{\mathrm{loglog}\,M(r)}{\log r}.$$

4. Next, let $Z:(z_1, z_2,\cdots\cdots)$ be a set of rational (positive, negative or zero) integers and $g(z)$ be a function, which takes integral values for all z_i. Denote by $\pi(n)$ the number of z_i's, whose absolute values are not greater than n, and using this $\pi(n)$ we form the functions $\Psi(x)$ and $\varphi(x)$ as in 1. Then we have:

Theorem B. *If we can chose a real function $\rho=\rho(r)$ such that*
$$(2r+1)\log 2r-2r-\int_0^r\varphi'(x)\log(\rho-x)dx+\log\rho(\rho-r)M(r)\longrightarrow-\infty \quad (B_1)$$
or
$$(2r+1)\log 2r-2r-\int_0^r(\varphi'(x)-1)\log(\rho^2-x^2)dx$$
$$+\log\rho(\rho-r)M(r)\longrightarrow-\infty \qquad (B_2)$$
and
$$\int_0^r\frac{\varphi(x)}{1+x}dx-\log M(r)\longrightarrow+\infty \ \text{as}\ r\to+\infty,$$
then $g(z)$ must be a polynomial.

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1) Skolem, Einige Sätze über ganzzahlige Lösungen gewisser Gleichungen und Ungleichungen, Math. Ann. 95 (1925), 1–68.

If $\varphi(x)$ is near $2x$, then the formula B_1 gives us more precise result than B, and if $\overline{\lim}\ \frac{\varphi(x)}{x} < 2$, then B is more useful than B_2. We can also get many special cases of this theorem as in §2.

5. We may also extend this problem to the case, where Z is a set of integral numbers in the corpus of the third or fourth root of unity[1].

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1) See Fukasawa, loc. cit.