102. *On the Theory of Surfaces in the Affine Space.*

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In connection to an interesting investigation on closed convex surfaces due to Prof. T. Kubota[1] the main object of this note is to treat the surfaces, called *affine moulding surfaces*, having ∞^1 system of curves which lie on parallel planes and are at the same time curves of contact of the enveloping cones with the surface. A special class of the affine moulding surfaces are defined as *affine surface of revolution.* For the latter we show that our definition is equivalent to that considered by Dr. Süss.[2]

1. The equation to the affine moulding surface may be easily deduced. In fact, take the curves on parallel planes ("parallel curves" say) and its conjugate system (" meridian curves " say) as parametric curves v, u; and let $(\xi(v), \eta(v), \zeta(v))$ be the curve (Γ-curve say) on which the vertices of the enveloping cones along $v=$const. lie. Then the surface is given by the equations:

$$\begin{aligned} x&=\exp\left(-\int\phi\, dv\right)\left(\int\xi\, \phi\exp\left(\int\phi\, dv\right)dv+U_1\right),\\ y&=\exp\left(-\int\phi\, dv\right)\left(\int\eta\, \phi\exp\left(\int\phi\, dv\right)dv+U_2\right),\\ z&=\exp\left(-\int\phi\, dv\right)\left(\int\zeta\, \phi\exp\left(\int\phi\, dv\right)dv+U_3\right),\end{aligned}$$

where $\phi=(a\xi+b\eta+c\zeta-v)^{-1}$, a, b, c being the (constant) direction cosines of the parallel planes; and U_1, U_2, U_3 are functions of u alone satisfying the relation

$$a\,U_1+b\,U_2+c\,U_3=0\,.$$

In what follows we will denote the surface in consideration by $[a, \xi, U]$.

Putting

$$T=\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix}, \qquad \overline{T}=\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix},$$

we can prove the formula

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1) T. Kubota, On the theory of closed convex surface, Proc. London Math. Soc. Ser. 2, **14** (1914); or Science Reports, Tohoku Imp. Univ. Ser. 1, **3** (1914).
2) W. Süss, Ein affingeometrische Gegenstück zu den Rotationsflächen, Math. Annalen, **98** (1928).

$$[a, T(\xi), U] \quad = T[\overline{T}(a), \xi, T^{-1}(U)].$$

Now we define the *affine deformability* of two surfaces with respect to the quadratic form in affine geometry in the same manner as that with respect to the line-element in elementary geometry. We can prove

Theorem. *When the Γ-curve is a space-curve, only the surface $[a, S(\xi), U]$ and those obtained from it by unit affinity are affine deformable to $[a, \xi, U]$, where*

$$S = \begin{pmatrix} 1+c_1a & c_1b & c_1c & k_1 \\ c_2a & 1+c_2b & c_2c & k_2 \\ c_3a & c_3b & 1+c_3c & k_3 \end{pmatrix},$$

c_1, c_2, c_3 and k_1, k_2, k_3 being arbitrary constants with the relations:

$$ac_1+bc_2+cc_3=0, \quad ak_1+bk_2+ck_3=0.$$

It is remarkable that S *is unimodular and forms an Abelian group.*

For the case where the Γ-curve is a plane curve, some analogous theorems may also be established.

Further we can prove

Theorem. *If the affine normal of the meridian curve of an affine moulding surface coincides with the affine surface-normal at the same point, then the surface must necessarily be a quadric.*

2. The affine surface of revolution is defined by the condition that the affine surface-normal of an affine moulding surface falls in the osculating plane of the meridian curves. This class is identical with those considered by Dr. Süss from quite a different standpoint, or in other words:

Theorem. *If the affine surface-normals of a surface all intersect a given straight line, then the surface must necessarily be an affine surface of revolution, and conversely.*

From our definition we can deduce the general equation to the affine surface of revolution from which many properties of the surface follow immediately. Moreover, we can deduce some characteristic properties of the surface in consideration. Among them the following seems to be somewhat interesting.

Theorem. *If one branch of Darboux's curves always lie on parallel planes, then the surface must necessarily be an affine surface of revolution.*

If the last theorem be taken as a definition of the affine surface of revolution, then we can also deduce the properties of these surfaces.