97 On Sufficient Conditions for the Uniqueness of the Solution of $\frac{dy}{dx}=f(x,y)$.

By Tatsujirô SHIMIZU.
Mathematical Institute, Tokyo Imperial University.

(Rec. July 4, 1928. Comm. by T. TAKAGI, M.I.A., July 12, 1928.)

We consider the differential equation

$$\frac{dy}{dx}=f(x,y)\,, \tag{1}$$

where $f(x,y)$ is a continuos function of x and y in the domain D $(0\leqq x\leqq a,\ |y|\leqq b)$. The equation (1) has always at least an integral curve which passes through $x=0$, $y=0$. For the uniqueness of the integral curve of (1) many sufficient conditions are known. Besides the well-known Lipschitz's condition $|f(x,y_1)-f(x,y_2)|<K|y_1-y_2|$, a sufficient condition

$$|f(x,y_1)-f(x,y_2)|<K|y_1-y_2|\log\frac{1}{|y_1-y_2|} \tag{2}$$

or more generally

$$|f(x,y_1)-f(x,y_2)|<\varphi(|y_1-y_2|),\ \text{where}\ \lim_{y=0}\int_\delta^y\frac{dy}{\varphi(y)}=-\infty, \tag{3}$$

was given by Osgood,[1] and another condition

$$|f(x,y_1)-f(x,y_2)|<k\frac{|y_1-y_2|}{x},\quad 0\leqq k<1, \tag{4}$$

by Rosenblatt.[2]

Recently Nagumo[3] without knowing Rosenblatt's condition (4) has discovered a more general condition

$$|f(x,y_1)-f(x,y_2)|<\frac{|y_1-y_2|}{x}\,. \tag{5}$$

Nagumo[4] and Perron[5] have extended the condition (5) to

$$|f(x,y_1)-f(x,y_2)|\leqq\frac{|y_1-y_2|}{x}\,. \tag{6}$$

Further Perron[6] has shown by simple examples that

---

1) Osgood, Monatshefte für Math. und Phys. **9** (1898) 331.
2) Rosenblatt, Arkiv för Mat. Astr. och Fys. **5** (1909) 2, 1.
3) Nagumo, Japanese Jour. of Math. **3** (1926) 107.
4) Nagumo, Japanese Jour. of Math. **4** (1927) 307.
5) Perron, Math. Zeitschr. **28** (1928) 216.
6) Perron. ibid.

$$|f(x, y_1)-f(x, y_2)| < (1+\varepsilon)\frac{|y_1-y_2|}{x}, \quad \varepsilon > 0 \tag{7}$$

can not be a sufficient condition.

On the other hand Montel[1] has given a general condition which implies (2), (3) and other conditions given by Tonelli and Bompiani. Recently Iyanaga[2] discovered a more general criterion for sufficient conditions which can be expressed as follows: In order that the equation (1) has in D a unique solution which passes through $x=0$, $y=0$ it is sufficient that we can find a differential equation

$$\frac{dv}{du}=g(u, v), \tag{8}$$

satisfying the following conditions:

1) $g(u, v)$ is defined in the domain D^* $(0 \leqq u \leqq a,\ 0 \leqq v \leqq 2b)$,

2) the equation (8) has at least one integral curve $v(u)=v(u, u_0, v_0)$ through any point (u_0, v_0) in $0 < u_0 \leqq a,\ 0 < v_0 \leqq 2b$, so that

2 a) $v(u)$ exists for $0 < u \leqq u_0$ and $0 \leqq v(u) \leqq 2b$,

and 2 b) $\lim_{u=0} v(u) > 0$ or $\lim_{u=0} v(u) = 0$ and $\lim_{u=0} \frac{dv}{du} > 0$,

3) For arbitrary y_1 and y_2 $(y_1 > y_2)$ in D we have the inequality

$$g(x, y_1-y_2) > f(x, y_1)-f(x, y_2).$$

The proof can be obtained as follows: Let $y_1(x)$ and $y_2(x)$ be two different solutions of (1) with $y_1(0)=y_2(0)=0$, then putting $y_1(x)-y_2(x)=\psi(x)$ we have $\psi(0)=0$ and $\psi'(0)=0$. Now suppose that there exist a point x_0, $0 < x_0 \leqq a$, at which $\psi(x_0) > 0$, and let $v(u)=v(u, u_0, v_0)$ be a solution of (8), where $x_0=u_0$, $\psi(x_0)=v_0$.

By 3) $v'(u)=g(u, v(u)) > f(u, y_1(u))-f(u, y_2(u))=y_1'(u)-y_2'(u)=\psi'(u)$.

By 2) $v(\varepsilon) > \psi(\varepsilon)$ for a sufficiently small ε. From $v(u_0)=v_0=\psi(u_0)$, we must have a point $\bar{u}$, $\bar{u} \leqq u_0$, such as $v(\bar{u})=\psi(\bar{u})$ and $v(\bar{u}-\delta) > \psi(\bar{u}-\delta)$.

Thus $$\lim_{\delta=0}\frac{v(\bar{u})-v(\bar{u}-\delta)}{\delta}=v'(\bar{u}) \leqq \psi'(\bar{u})=\lim_{\delta=0}\frac{\psi(\bar{u})-\psi(\bar{u}-\delta)}{\delta},$$

which contradicts $v'(u) > \psi'(u)$.

This Iyanaga's criterion is of very general character, from which all the sufficient conditions above cited can be deduced. Here I will give some new particular conditions, which seem not without interest.

Theorem: For the uniqueness of the solution of (1) each of the following conditions is sufficient.

1) Montel, Bull. Scie. Math. France **50** (1926) 215.
2) This will appear in Japanese Jour. of Math. **5** (1928).

Condition I.[1)]

$$|f(x, y_1)-f(x, y_2)|<(1+\varepsilon(x))\frac{|y_1-y_2|}{x}, \text{ where } \varepsilon(x)>0$$

and $\lim_{x=0}\int_\delta^x \frac{\varepsilon(x)}{x}dx>-M,\ 0<M<\infty,\ \delta>0.$

Condition II.[2)]

$$|f(x, y_1)-f(x, y_2)|<\alpha\frac{|y_1-y_2|}{x}+\beta|y_1-y_2|\log\frac{1}{|y_1-y_2|},$$

$0\leqq\alpha<1,\quad 0\leqq\beta.$

Condition III.[3)]

$$|f(x, y_1)-f(x, y_2)|<\frac{|y_1-y_2|}{x}\frac{\left|\log\frac{1}{|y_1-y_2|}\right|^l}{\left|\log\frac{1}{x}\right|^k},\quad 0<l<k,\ l\leqq 1.$$

Proof. For the proof of Cond. I we may apply Iyanaga's criterion and consider the differential equation

$$\frac{dv}{du}=(1+\varepsilon(u))\frac{v}{u}=g(u, v) \tag{9}$$

and put an indefinite integral $\int\frac{\varepsilon(u)}{u}du=I(u)$.

The general solution (9) is $v=u\,e^{C+I(u)}$, C being an integration-constant. By $I(u)>-M$ we have for any $C\neq 0,\ \neq\infty$,

$$\lim_{u=0}\frac{dv}{du}=e^{C+I(u)}+\varepsilon(u)e^{C+I(u)}>0.$$

Hence Cond. I is proved.

For the proof of Cond. II we consider

$$\frac{dv}{du}=\alpha\frac{v}{u}-\beta v\log v. \tag{10}$$

The general solution of (10) is $\left(\text{putting an indefinite integral} \int\frac{e^{\beta u}}{u}du\right.$ $\left.=G(u)\right),$ $v=e^{\alpha e^{-\beta u}G(u)+C e^{-\beta u}}$.

On such curves $v(u, C)$ we have

$$\log\frac{dv}{du}=\alpha e^{-\beta u}G(u)+Ce^{-\beta u}+\log\left(-\alpha\beta e^{-\beta u}G(u)+\frac{\alpha}{u}-\beta Ce^{-\beta u}\right)$$

1) Compare with (6) and (7).
2) Compare with (2) and (3).
3) From Cond. III we can obtain a sharper condition than Cond. I and (6), for example, for $y\leqq x^{(-\log x)^m}$, $m>\frac{k-l}{l}$, Cond. III and for $y>x^{(-\log x)^m}$ Cond. I.

$$=\alpha e^{-\beta u}G(u)+\log\frac{1}{u}+O(1).$$

Putting $\alpha=\frac{1}{1+\gamma}$, $\gamma>0$, and choosing $\varepsilon<\gamma$ and then $\delta(\varepsilon)$ so that

$$\log\frac{\delta(\varepsilon)}{u}<G(\delta(\varepsilon))-G(u)<(1+\varepsilon)\log\frac{\delta(\varepsilon)}{u},$$

we have

$$\log\frac{dv}{du}=\left(1-\frac{1+\varepsilon}{1+\gamma}e^{-\beta u}\right)\log\frac{1}{u}+O(1)\to+\infty.$$

Hence
$$\lim_{u=0}\frac{dv}{du}=+\infty.$$

For the proof of Cond. III we consider

$$\frac{dv}{du}=\frac{v}{u}\frac{(-\log v)^l}{(-\log u)^k}. \tag{11}$$

The general solution of (11) is, for $0<l<k<1$,

$$v=e^{-\left\{\frac{1-l}{1-k}(-\log u)^{1-k}+C\right\}^{\frac{1}{1-l}}}.$$

On such curves $v(u, C)$ we have by $l<k$

$$\log\frac{dv}{du}=-\left\{\frac{1-l}{1-k}(-\log u)^{1-k}+C\right\}^{\frac{1}{1-l}}+\log(-\log u)^{-k}$$
$$+\log\left\{\frac{1-l}{1-k}(-\log u)^{1-k}+C\right\}^{\frac{l}{1-l}}+\log\frac{1}{u}\to+\infty.$$

Hence
$$\lim_{u=0}\frac{dv}{du}=+\infty.$$

Similarly for $l<k=1$ and $1=l<k$.

Remark: During the preparation for this paper I was told that Mr. Fukuhara[1)] had also given a sufficient condition

$$|f(x,y_1)-f(x,y_2)|<k(x)|y_1-y_2|,\ \text{where}\ \lim x e^{-\int_\delta^x k(x)dx}<M,$$

which is identical with Cond. I, for, putting $k(x)=\frac{1+\varepsilon(x)}{x}$ we have $\int\frac{\varepsilon(x)}{x}dx>-M$, and conversely, putting $\frac{\varepsilon(x)+1}{x}=k(x)$, we have $\lim_{x=0} x e^{-\int k(x)dx}<M$.

---

1) This will appear in Japanese Jour. of Math. **5** (1928).