27. On a Property of the Fourier Series of an Almost Periodic Function.

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In the theory of an analytic almost periodic function Bohr's "Randwertsatz" plays an important rôle. It concerns with an analytic almost periodic function whose Dirichlet exponents have the same sign. In this paper, we shall prove a theorem which is regarded as an extension of Randwertsatz to the case where Dirichlet exponents have not the same sign.[1)]

Theorem. If an indefinite integral of an almost periodic function of a real variable

$$f(x) \sim \sum A_n e^{i\Lambda_n x} \qquad (-\infty < x < \infty)$$

is bounded, then the series

$$\sum A_n \operatorname{sgn} \Lambda_n \cdot e^{-\sigma|\Lambda_n|} e^{i\Lambda_n x},$$

where σ is any positive number, is the Fourier series of an almost periodic function.

For the proof,[2)] consider the function

$$\varphi_A(t) = \frac{1}{\pi}\int_0^A \frac{xf(x+t) - xf(-x+t)}{\sigma^2 + x^2}\,dx \qquad (-\infty < t < \infty),$$

where A and σ are any positive numbers. Then $\varphi_A(t)$ is an almost periodic function. Indeed, taking τ as a translation-number of $f(t)$ belonging to ε, i.e.

$$|f(t+\tau) - f(t)| \leqq \varepsilon, \qquad -\infty < t < \infty,$$

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1) In case where Dirichlet exponents have not the same sign, the following theorem is known:

If $\quad f(s) \sim \Sigma A_n e^{\Lambda_n s}, \qquad s = \sigma + it$

is almost periodic in $\langle \alpha, \beta \rangle$ and if its integral $F(s)$ is also almost periodic in $\langle \alpha, \beta \rangle$ (which is true if $F(s)$ is bounded), then the series

$$\sum_{\Lambda_n < 0} A_n e^{\Lambda_n s}, \qquad \sum_{\Lambda_n > 0} A_n e^{\Lambda_n s}$$

are the Dirichlet series of two functions $f_1(s)$, almost periodic in $\langle \alpha, +\infty)$ and $f_2(s)$, almost periodic in $(-\infty, \beta\rangle$.

2) I owe this method of proof to Mr. Favard's paper: Sur la fonction conjuguée d'une fonction presque-périodique, Matematisk Tidsskrift (1934), p. 57.

we have:

$$\varphi_A(t+\tau)-\varphi_A(t)$$
$$=\frac{1}{\pi}\int_0^A \frac{x[f(x+t+\tau)-f(x+t)]-x[f(-x+t+\tau)-f(-x+t)]}{\sigma^2+x^2}dx$$

from which

$$|\varphi_A(t+\tau)-\varphi_A(t)|\leqq\frac{\varepsilon}{\pi}\log\left(1+\frac{A^2}{\sigma^2}\right).$$

Thus a translation-number τ of $f(t)$ belonging to ε is a translation-number of $\varphi_A(t)$ belonging to $\varepsilon\pi^{-1}\log(1+A^2\sigma^{-2})$.

Now consider the Fourier series of $\varphi_A(t)$. We have

$$a_A(\lambda)=M\{\varphi_A(t)e^{-i\lambda t}\}$$
$$=\lim_{T\to\infty}\frac{1}{\pi}\int_0^A\frac{xdx}{\sigma^2+x^2}\left\{\frac{1}{T}\int_0^T[f(x+t)-f(-x+t)]e^{-i\lambda t}dt\right\}$$

and

$$\frac{1}{T}\int_0^T[f(x+t)-f(-x+t)]e^{-i\lambda t}dt$$
$$=e^{i\lambda x}\cdot\frac{1}{T}\int_x^{x+T}f(t)e^{-i\lambda t}dt-e^{-i\lambda x}\cdot\frac{1}{T}\int_{-x}^{-x+T}f(t)e^{-i\lambda t}dt.$$

When T increases infinitely, it is well known that

$$\frac{1}{T}\int_x^{x+T}f(t)e^{-i\lambda t}dt,\qquad \frac{1}{T}\int_{-x}^{-x+T}f(t)e^{-i\lambda t}dt$$

tend uniformly to the same limit $a(\lambda)$.

Thus for any positive number ε, one can find an integer m such that for $T\geqq m$ the following inequalities hold:

$$\frac{1}{T}\int_x^{x+T}f(t)e^{-i\lambda t}dt=a(\lambda)+\varepsilon'(x,T)\quad\text{with}\quad|\varepsilon'(x,T)|\leqq\varepsilon;$$

$$\frac{1}{T}\int_{-x}^{-x+T}f(t)e^{-i\lambda t}dt=a(\lambda)+\varepsilon''(x,T)\quad\text{with}\quad|\varepsilon''(x,T)|\leqq\varepsilon,$$

$$\left|a_A(\lambda)-\frac{1}{\pi}\int_0^A\frac{xdx}{\sigma^2+x^2}\left\{\frac{1}{T}\int_0^T[f(x+t)-f(-x+t)]e^{-i\lambda t}dt\right\}\right|\leqq\varepsilon.$$

Therefore we have

$$\frac{1}{\pi}\int_0^A\frac{xdx}{\sigma^2+x^2}\left\{\frac{1}{T}\int_0^T[f(x+t)-f(-x+t)]e^{-i\lambda t}dt\right\}$$
$$=\frac{2ia(\lambda)}{\pi}\int_0^A\frac{x\sin\lambda x}{\sigma^2+x^2}dx+\frac{1}{\pi}\int_0^A\frac{x\varepsilon'(x,T)e^{i\lambda x}-x\varepsilon''(x,T)e^{-i\lambda x}}{\sigma^2+x^2}dx,$$

from which

$$\left|a_A(\lambda)-\frac{2ia(\lambda)}{\pi}\int_0^A\frac{x\sin\lambda x}{\sigma^2+x^2}dx\right|\leqq\varepsilon\left\{1+\frac{1}{\pi}\log\left(1+\frac{A^2}{\sigma^2}\right)\right\}.$$

But we may take ε as small as we please; it immediately follows

$$a_A(\lambda)=\frac{2ia(\lambda)}{\pi}\int_0^A\frac{x\sin\lambda x}{\sigma^2+x^2}dx$$

so that

$$\varphi_A(t)\sim\frac{2i}{\pi}\sum_{n=1}^{\infty}\left\{A_n\int_0^A\frac{x\sin\Lambda_n x}{\sigma^2+x^2}dx\right\}e^{i\Lambda_n t}.$$

Let $F(t)$ be an indefinite integral of $f(t)$. Under the hypothesis that $F(t)$ is bounded for $-\infty<t<\infty$, it is easily seen that when A increases infinitely, $\varphi_A(t)$ converges uniformly to the limit function $\varphi(t)$ which is itself an almost periodic function. Indeed, taking two positive numbers A_1 and A_2 $(A_1<A_2)$, we have:

$$\begin{aligned}&\varphi_{A_2}(t)-\varphi_{A_1}(t)\\&=\frac{1}{\pi}\left[\frac{xF(x+t)+xF(-x+t)}{\sigma^2+x^2}\right]_{A_1}^{A_2}-\frac{1}{\pi}\int_{A_1}^{A_2}\frac{(\sigma^2-x^2)[F(x+t)+F(-x+t)]}{(\sigma^2+x^2)^2}dx.\end{aligned}$$

Let M be the upper boundary of $|F(t)|$. Then one can deduce from the above equality the following inequality:

$$|\varphi_{A_2}(t)-\varphi_{A_1}(t)|\leqq\frac{4M}{\pi}\cdot\frac{A_1}{\sigma^2+A_1^2}<\frac{4M}{\pi A_1}.$$

This shows that $\varphi_A(t)$ converges uniformly to the limit function $\varphi(t)$ when A increases infinitely.

On the other hand, it is well known that the Fourier series of the limit function of uniformly convergent sequence of almost periodic functions is the limit of the Fourier series of the almost periodic functions in consideration. Therefore we have

$$\varphi(t)\sim\frac{2i}{\pi}\sum_{n=1}^{\infty}\left\{A_n\int_0^{\infty}\frac{x\sin\Lambda_n x}{\sigma^2+x^2}dx\right\}e^{i\Lambda_n t}.$$

But one can easily assure that the following equality holds:

$$\int_0^{\infty}\frac{x\sin\Lambda_n x}{\sigma^2+x^2}dx=\frac{\pi}{2}\operatorname{sgn}\Lambda_n\cdot e^{-\sigma|\Lambda_n|}.$$

Then we have

$$\varphi(t)\sim i\sum_{n=1}^{\infty}A_n\operatorname{sgn}\Lambda_n\cdot e^{-\sigma|\Lambda_n|}e^{i\Lambda_n t}.$$

Thus our theorem is completely proved.