111. A Note on the Continuous Representation of Topological Groups.

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§ I. In a recent paper,[1] the author treated the group $\mathfrak{D}$ embedded in the *metrical complete ring* $\mathfrak{R}$. Such a group $\mathfrak{D}$ is, by [I], a *Lie group* (as defined in [I][2]) if and only if $\mathfrak{D}$ is locally compact.

In another paper,[3] I obtained a result saying that, if $\mathfrak{D}$ is continuously homomorphic to a connected and locally bicompact topological group $\mathfrak{G}$ (*without any countability axiom*), then $\mathfrak{D}$ is a *Lie representation* (defined in [II]).

A Lie representation $\mathfrak{D}$ is not necessarily a Lie group as defined in [I] (see [II]), though the *infinitesimal operators* of $\mathfrak{D}$ obey the customary rule of the ordinary *Lie-ring,* when $\mathfrak{G}$ is a Lie group (see [II]).

However we may add the following remark:

The representation $\mathfrak{D}$ *of* $\mathfrak{G}$ *is a Lie group* (*as defined in* [I]) *if and only if the homomorphic mapping* $\mathfrak{G}\to\mathfrak{D}$ *is open.*

Here a continuous mapping is called *open* if the mapped image of any open set is an open set.

Proof. $\mathfrak{D}$ is isomorphic to the quotient group $\mathfrak{G}/\mathfrak{N}$, where $\mathfrak{N}$ is an invariant subgroup closed in $\mathfrak{G}$. We call any set in $\mathfrak{G}/\mathfrak{N}$ open if and only if it corresponds to an open set in $\mathfrak{G}$ by the homomorphic mapping $\mathfrak{G}\to\mathfrak{G}/\mathfrak{N}$. Then $\mathfrak{G}/\mathfrak{N}$ is connected and locally bicompact with $\mathfrak{G}$. Thus $\mathfrak{D}$ is continuously isomorphic to $\mathfrak{G}/\mathfrak{N}$ and the mapping $\mathfrak{G}\to\mathfrak{D}$ is open if and only if the mapping $\mathfrak{G}/\mathfrak{N}\to\mathfrak{D}$ is open.

Hence we may- and shall- assume that $\mathfrak{D}$ is continuously isomorphic to $\mathfrak{G}$.

Thus if the mapping $\mathfrak{G}\to\mathfrak{D}$ is open, $\mathfrak{G}$ and $\mathfrak{D}$ are homeomorphic with each other, and hence $\mathfrak{D}$ is locally bicompact and connected with $\mathfrak{G}$. This proves the sufficiency of the condition of the remark.

As the group $\mathfrak{D}$ is embedded in $\mathfrak{R}$, $\mathfrak{D}$ does not contain an arbitrarily small cyclic subgroup ($\neq$ identical group[4]). $\mathfrak{G}$ enjoys the same property, for $\mathfrak{D}$ is continuousy isomorphic to $\mathfrak{G}$. By a theorem due to A. Komatu and S. Kakutani (see [II]) $\mathfrak{G}$ satisfies the first axiom of countability, since $\mathfrak{G}$ is locally bicompact. Hence $\mathfrak{G}$ is metrisable by a result of S. Kakutani.[5]

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1) K. Yosida: On the group embedded in the metrical complete ring, Jap. J. of Math. **13** (1936). This paper will be cited as [I].

2) The condition γ) in the definition of a *Lie group* in [I] is not an essential one, it states that the group is connected. Hence it may be omitted out of the definition.

3) K. Yosida: On the group embedded in the metrical complete ring, II, to appear soon in Jap. J. of Math. This paper will be cited as [II].

4) For the proof see [I].

5) S. Kakutani: Über die Metrisation der topologischen Gruppen, Proc. **12** (1936). Cf. also G. Birkhoff: A note on topological groups, Comp. Math. **3** (1936).

Thus $\mathfrak{G}$ is locally separable and connected. We see that such a group is separable, by applying Schreier's theorem[1] on connected groups.

Hence by H. Freudenthal's[2] result, the mapping $\mathfrak{G} \rightarrow \mathfrak{D}$ is open if $\mathfrak{D}$ is locally compact, for then $\mathfrak{D}$ is a Lie group by [I].

§ II. With regards to the dimension relation by the continuous homomorphic mapping $\mathfrak{G} \rightarrow \mathfrak{D}$ we may prove the following remark:

If $\mathfrak{G}$ is compact and separable,[3] we have

$$(d) \qquad \dim \mathfrak{D} \leqq \dim \mathfrak{G},$$

without assuming the group $\mathfrak{D}$ to be embedded in $\mathfrak{R}$. It may be any topological group.

Proof. $\mathfrak{D}$ is compact and separable with $\mathfrak{G}$, and hence the mapping $\mathfrak{G} \rightarrow \mathfrak{D}$ is open.[4] Thus $\mathfrak{D}$ is topologically isomorphic to $\mathfrak{G}/\mathfrak{N}$, where $\mathfrak{N}$ denotes an invariant subgroup closed in $\mathfrak{G}$.[5]

By a theorem of H. Freudenthal,[6] there exists a decreasing sequence $\{\mathfrak{H}_m\}$ of closed invariant subgroups in $\mathfrak{G}$, $\lim\limits_{m\to\infty} \mathfrak{H}_m = e$ (unit element of $\mathfrak{G}$), such that $\mathfrak{G}/\mathfrak{H}_m$ $(m=1, 2, \ldots)$ is topologically isomorphic to a compact matrix group (Lie group by [I]). $\mathfrak{G}$ is thus *G_n-adic generated*[7] by the sequence $\{\mathfrak{G}/\mathfrak{H}_m\}$, and $\dim \mathfrak{G} = \lim\limits_{m\to\infty} \dim \mathfrak{G}/\mathfrak{H}_m$.[8]

As $\mathfrak{H}_m \supseteqq \mathfrak{H}_{m+1}$, $\lim \mathfrak{H}_m = e$, it is easy to see that the group $\mathfrak{G}/\mathfrak{N}$ and $\mathfrak{N}$ are G_n-adic generated by the sequences

$$\left\{\mathfrak{G}/\mathfrak{N}\Big/\mathfrak{N}\mathfrak{H}_m/\mathfrak{N}\right\} \quad \text{and} \quad \left\{\mathfrak{N}/\mathfrak{N}V\mathfrak{H}_m\right\}^{9)}$$

respectively. $\mathfrak{N}\mathfrak{H}_m/\mathfrak{H}_m$ is a compact matrix group (Lie group by [I]), since it is a closed invariant subgroup in the compact matrix group $\mathfrak{G}/\mathfrak{H}_m$. Thus the topological isomorphisms

$$\begin{cases} \mathfrak{G}/\mathfrak{N}\mathfrak{H}_m \cong \mathfrak{G}/\mathfrak{H}_m\Big/\mathfrak{N}\mathfrak{H}_m/\mathfrak{H}_m \cong \mathfrak{G}/\mathfrak{N}\Big/\mathfrak{N}\mathfrak{H}_m/\mathfrak{N}, \\ \mathfrak{N}\mathfrak{H}_m/\mathfrak{H}_m \cong \mathfrak{N}/\mathfrak{N}V\mathfrak{H}_m{}^{10)} \end{cases}$$

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1) O. Schreier: Abstrakte kontinuierliche Gruppen, Hamburg Abh. Math. Sem. **4** (1925).

2) H. Freudenthal: Einige Sätze über topologische Gruppen, Ann. of Math. **37** (1936), p. 47. This paper will be cited as FI.

3) The separability hypothesis may be replaced by the first axiom of countability, for then $\mathfrak{G}$ is metrisable by Kakutani's theorem, loc. cit. When $\mathfrak{G}$ is locally compact, connected, separable and zero-dimensional, (d) is obtained by H. Freudenthal, loc. cit. p. 51.

4) FI, p. 47.

5) FI, p, 49. The topology in quotient group is defined as in § I.

6) H. Freudenthal: Topologische Gruppen mit genügend vielen fastperiodischen Funktionen, Ann. of Math. **37** (1936). This paper will be cited as FII. Cf. also E. R. van Kampen: Almost periodic functions and compact groups, Ann. of Math. **37** (1936).

7) FII, p. 69.

8) FII, p. 71.

9) $\mathfrak{N}\mathfrak{H}$ denotes the set of all the products nh, where $n \in \mathfrak{N}$, $h \in \mathfrak{H}$. $\mathfrak{N}V\mathfrak{H}$ denotes the set of all the elements common to $\mathfrak{N}$ and $\mathfrak{H}$.

10) FI, p. 50.

show that $\mathfrak{N}/\mathfrak{N}V\mathfrak{H}_m$ and $\mathfrak{G}/\mathfrak{N}\mathfrak{H}_m$ are compact Lie groups, and hence we have[1)]

$$\dim \mathfrak{G}/\mathfrak{N} = \lim_{m\to\infty} \dim \mathfrak{G}/\mathfrak{N}\mathfrak{H}_m\,, \qquad \dim \mathfrak{N} = \lim_{m\to\infty} \mathfrak{N}/\mathfrak{N}V\mathfrak{H}_m\,.$$

By considering the *canonical parameters* we obtain

$$\dim \mathfrak{G}/\mathfrak{N}\mathfrak{H}_m = \dim \mathfrak{G}/\mathfrak{H}_m \Big/ \mathfrak{N}\mathfrak{H}_m/\mathfrak{H}_m = \dim \mathfrak{G}/\mathfrak{H}_m - \dim \mathfrak{N}\mathfrak{H}_m/\mathfrak{H}_m\,,$$

and thus[2)]

$$(d')\quad \dim \mathfrak{G}/\mathfrak{N} = \lim_{m\to\infty} \dim \mathfrak{G}/\mathfrak{H}_m - \lim_{m\to\infty} \dim \mathfrak{N}/\mathfrak{N}V\mathfrak{H}_m = \dim \mathfrak{G} - \dim \mathfrak{N}\,.$$

This proves (d) by $\mathfrak{D} \cong \mathfrak{G}/\mathfrak{N}$.

Q. E. D.

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1) FII, p. 71.

2) After this paper is completed the author found that (d') was also obtained by van Kampen in somewhat another way. E. R. van Kampen: A note on a theorem by Pontrjagin, Amer. J. of Math. 51, **1** (1936) p. 178.