26. The Multiplication-Theorems of the Cauchy Series.

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The chief object of this note is to establish the multiplication-theorems[1] of Cauchy series under the conditions which are in some respects more general than those given in our previous paper. Their connections with certain aspects of an interpolation will also be indicated in § 3.

§ 1. We consider a linear functional $l_\xi\{f(\xi)\}\equiv\int_0^b f(\xi)\,d\varphi(\xi)$ associated to a given function $\varphi(\xi)$ of bounded variations over a finite interval $(0, b)$. To each function $f(x)$, which is Lebesque-integrable over $(0, b)$, we shall correspond a sequence of contour-integrals[2] defined by

$$(1)\qquad f_r(x)\equiv S_{\mathfrak{C}_r}(x;f)\equiv\frac{1}{2\pi i}\oint_{\mathfrak{C}_r}\frac{e^{\lambda x}}{G(\lambda)}\,l_\xi\left\{e^{\lambda\xi}\int_0^\xi e^{-\lambda\eta}f(\eta)\,d\eta\right\}d\lambda\,,$$

where the integral function $G(\lambda)\equiv l_\xi\{e^{\lambda\xi}\}$.[3] We observe

Theorem I. Consider a function $f(x)$ belonging to $L^p(0, b)$[4] *with $p>1$. If there is a sequence $\{S_{\mathfrak{C}_r}(x;f)\}$ $(r=1, 2, 3, \ldots\ldots)$ such that*

$$(2)\qquad \lim_{r\to\infty}\int_0^b |f(x)-S_{\mathfrak{C}_r}(x;f)|^p\,dx=0\,,$$

then, for any given function $g(x)$ belonging to $L^q(0, b)$, where $1/p+1/q=1$, we have

$$(3)\qquad (f, g)\equiv l_\xi\left\{\int_0^\xi f(\xi-\eta)g(\eta)\,d\eta\right\}$$
$$=\lim_{r\to\infty}\frac{1}{2\pi i}\oint_{\mathfrak{C}_r}\frac{l_\xi\left\{e^{\lambda\xi}\int_0^\xi e^{-\lambda\eta}f(\eta)\,d\eta\right\}l_\xi\left\{e^{\lambda\xi}\int_0^\xi e^{-\lambda\eta}g(\eta)\,d\eta\right\}}{G(\lambda)}\,d\lambda\,.$$

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1) *T. Kitagawa*: On the theory of linear translatable functional equation and Cauchy's series. Jap. Journ. Math., **XIII** (1937) pp. 233–332. Cf. specially Chap. III, §§ 12–14. We shall note this paper by [T].

2) In this note as well as in [T], a sequence of contours $\{\mathfrak{C}_r\}$ $(r=1, 2, 3, \ldots\ldots)$ is always assumed to be selected such that (i) $\mathfrak{C}_r$ is contained in the domain enclosed by $\mathfrak{C}_{r+1}$ and, $\mathfrak{C}_r$ diverges to the whole plane as $r\to\infty$, that is, the distance between the origin and $\mathfrak{C}_r$, which we denote by d_r, tends to infinity as $r\to\infty$; (ii) there are merely two points of the intersections of $\mathfrak{C}_r$ with the imaginary axis of λ-plane, which we denote by $i\rho_r$ and $-i\rho_r$ respectively. The interval $(0, b)$ is assumed to be closed.

3) In [T] we have considered a $\mathfrak{C}$-section of Cauchy series with respect to a linear translatable operation. The present form can be recognised as a special case of the former one, as we can consider a linear translatable operation defined by $l_\xi\{f(x+\xi)\}=\int_0^b f(x+\xi)\,d\varphi(\xi)$. Cf. Also *J. Delsarte*: Les fonctions "moyenne-périodiques." Journ. d. Math. Pures et appliquées, neuvième série, tome quatorziéeme (1935).

4) $L^p(0, b)$ denotes the class of all the functions $f(x)$ which are defined over $(0, b)$ and $|f(x)|^p$ are integrable in the sense of Lebesque.

Proof: By the linearity of the functional l and the Hölder's inequality we obtain

$$|(f,g)-(f_r,g)| \leqq \int_0^b \left\{\int_0^\xi |f(\xi-\eta)-f_r(\xi-\eta)|^p d\eta\right\}^{\frac{1}{p}} \left\{\int_0^\xi |g(\eta)|^q d\eta\right\}^{\frac{1}{q}} |d\varphi(\xi)|$$

$$\leqq \left\{\int_0^b |f(\eta)-f_r(\eta)|^p d\eta\right\}^{\frac{1}{p}} \left\{\int_0^b |g(\eta)|^q d\eta\right\}^{\frac{1}{q}} \int_0^b |d\varphi(\xi)|.$$

This leads us to $\lim_{r\to\infty} (f_r, g)=(f,g)$, which is equivalent to (3). q. e. d.

In the right-hand side of (3) the contour-integral with respect to each $\mathfrak{C}_r$ can be easily expressed by the formal calculation as the bilinear form[1] of the expansion-coefficients of the Cauchy series of $f(x)$ and $g(x)$. Further we shall give

Example 1. If all the zero-points of $G(\lambda)$ are simple and imaginary, and if, denoting them by $\{i\lambda_n\}$ $(n=0, \pm 1, \pm 2, \ldots\ldots)$, they are subjected to the condition that there is a constant D with

$$(4) \qquad |\lambda_n - n| \leqq D < \frac{1}{\pi^2} \qquad (n=0, \pm 1, \pm 2, \ldots\ldots),$$

then the hypothesis to Theorem I holds for any function belonging to $L^2(0, 2\pi)$. To see this, we have only to notice that under the assumption (4) our Cauchy expansion coincides with the non-harmonic Fourier series[2] with respect to the function set $\{e^{i\lambda_n}\}$[3] $(n=0, \pm 1, \pm 2, \ldots\ldots)$.

§ 2. Now we turn to the consideration of the functions which are defined over $(-\infty, \infty)$ and Lebesque-integrable in any finite interval. We shall consider a sequence of contour-integrals

$$(5) \qquad S_{\mathfrak{C}_r}(x, x_0; f) \equiv \frac{1}{2\pi i} \oint_{\mathfrak{C}_r} \frac{e^{\lambda x}}{G(\lambda)} l_\xi \left\{ e^{\lambda\xi} \int_0^\xi e^{-\lambda\eta} f(\eta)\, d\eta \right\} d\lambda,$$

for each fixed x_0 in $-\infty < x_0 < \infty$. We adopt

Definition 1. A Cauchy series of $f(x)$ which is defined by the sequence (5) is said *to be of Fourier type with respect to the sequence of the contours* $\{\mathfrak{C}_r\}$ $(r=1, 2, 3\ldots\ldots)$, if, for each fixed x_0 in $-\infty < x_0 < \infty$,

$$(6) \qquad S_{\mathfrak{C}_r}(x, x_0; f) - \frac{1}{\pi} \int_{x_0}^{x_0+b} f(t) \frac{\sin \rho_r(x-t)}{x-t} dt$$

tends uniformly to zero over any interval $(x_0+\varepsilon, x_0+b-\varepsilon)$, ε being arbitrarily given positive number, as r tends to infinity.

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1) Cf. [T], Chap. III, § 13, Multiplication-theorems.

2) See *Paley and Wiener*: Fourier transforms in the complex domain. Am. Math. Coll. Publ. XIX, Chap. VII, § 29, Theorem XXXVII (p. 100) and § 30, (pp. 108–113). Specially the relation (29.02) is important for the present purpose, when $g_n(x) = e^{i\lambda_n x}$. The choice of a sequence of contours $\{\mathfrak{C}_r\}$ will be indicated in the following paragraph, Example 2.

3) The proof will be given in the author paper: The Parseval theorem of Cauchy series and the inner products of certain Hilbert spaces. The paper will appear elsewhere.

We shall prove

Theorem II. Consider a function $f(x)$ which is L^p-integrable in any finite interval. Let it be a solution of the functional equation

$$(7)\qquad \Lambda f(x)\equiv\int_0^b f(x+t)\,d\varphi(t)=0\,,\qquad (-\infty<x<\infty)\,.$$

Then for any contour $\mathfrak{C}$, $S_{\mathfrak{C}}(x, x_0; f)$ is independent of x_0; that is, for each x_0 in $-\infty<x_0<\infty$, $S_{\mathfrak{C}}(x, x_0; f)=S_{\mathfrak{C}}(x, 0; f)$.

If furthermore the Cauchy series of $f(x)$ is of Fourier type with respect to certain sequence $\{\mathfrak{C}_r\}$, then for any given finite interval (α, β), we have

$$(8)\qquad \lim_{r\to\infty}\int_\alpha^\beta |f(x)-S_{\mathfrak{C}_r}(x, 0; f)|^p dx=0\,,$$

and consequently the Hypothesis to Theorem I and hence the Conclusion remain true.

Proof: The first part[1] of the Theorem was proved in my previous paper. In virtue of the assumption that the Cauchy series of $f(x)$ is of the Fourier type, it is evident that, for any given x_0 and any given small positive number δ, we have

$$(9)\qquad \lim_{r\to\infty}\int_{x_0+\delta}^{x_0+b-\delta} |f(x)-S_{\mathfrak{C}_r}(x, x_0; f)|^p dx=0\,.$$

Now let (α, β) be an arbitrarily given finite interval. By a suitable choice of the sequence $\{x_0^{(i)}\}$ $(i=1, 2, \ldots\ldots, N)$, we may cover[2] the interval (α, β) by the sequence of the intervals $\{(x_0^{(i)}+\delta, x_0^{(i)}+b-\delta)\}$ $(i=1, 2, \ldots\ldots, N)$. In view of the relation $S_{\mathfrak{C}}(x, x_0^{(i)}; f)=S_{\mathfrak{C}}(x, 0; f)$, we observe

$$\begin{aligned}(10)\qquad \int_\alpha^\beta |f(x)-S_{\mathfrak{C}_r}(x, 0; f)|^p dx &\leqq \sum_{i=1}^N \int_{x_0^{(i)}+\delta}^{x_0^{(i)}+b-\delta} |f(x)-S_{\mathfrak{C}_r}(x, 0; f)|^p dx \\ &= \sum_{i=1}^N \int_{x_0^{(i)}+\delta}^{x_0^{(i)}+b-\delta} |f(x)-S_{\mathfrak{C}_r}(x, x_0^{(i)}; f)|^p dx\,.\end{aligned}$$

In combination of (9) and (10), we reach (8). q. e. d.

Example 2. By the theorem of N. Levinson[3] on the non-harmonic Fourier series, it is readily seen that, if, in the hypothesis to Example 2, the condition (4) is replaced by: $|\lambda_n-n|\leqq D<(p-1)/2p$,[4] for all n, then the Cauchy series of functions which are L^p-integrable in any finite interval are of the Fourier type in our terminology, when the

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1) Cf. [T], Chap. II, § 6, Theorem VI, I and its proof.

2) By the covering theorem of Heine-Borel. The same method is also used in the proof of Theorem VI, II in [T].

3) *N. Levinson*: On Non-Harmonic Fourier Series, Ann. Math. **37** (1936) pp. 917–936. Cf. specially Theorem I in p. 920.

4) In this condition we can prove that our Cauchy expansion is a non-harmonic Fourier series. The proof will be given in a paper cited on the footenote (3) in p. 97.

sequence of contours $\{\mathfrak{C}_r\}$ is assumed to be so chosen that each $\mathfrak{C}_r$ contains all the first $2r+1$ zero-points, i. e., $\{i\lambda_k\}$ $(k=0, \pm 1, \ldots\ldots, \pm 2r)$ in its interior, the others remaining in the exterior of $\{\mathfrak{C}_r\}$.

Example 3. In the terminology of our previous paper [T], we may enunciate the following: if there is a sequence of contours $\{\mathfrak{C}_r\}$ $(r=1, 2, 3, \ldots\ldots)$ such that, for any given $\delta>0$, $\mathfrak{C}_r \varepsilon \mathfrak{C}(\delta, -\delta)$[1] and further that with respect to this sequence and each δ, $G(\lambda)=\mathfrak{C}_r - O\,(0; \delta, b-\delta)$,[2] then the Cauchy series of functions which are Lebesgue-integrable in any finite interval are of the Fourier type with respect to this sequence of the contours $\{\mathfrak{C}_r\}$.[3]

§ 3. We shall finish this note with a remark to the connections with the multiplication-theorem of Cauchy series and the interpolation-theory. The cardinal series and its connection with certain aspects of the theory of Fourier series and integrals were thoroughly investigated by several authors of the Edinburgh school. The cardinal series concerns itself with the interpolation at all integers where the values of a function to be interpolated are assigned. From the general standpoint of the interpolation theory it may be desirable to generalise their ingenious method to a more general interpolation at the irregularly distributed points, where, in general, not only the values of a function, but also the values of their derivatives up to certain respective orders are assigned to be interpolated. For this purpose we wish to commounicate the following theorem.

Assumption I. An integral function $C(z)$ is definied by

$$C(z)=l_\xi\left\{\int_0^\xi e^{z(\xi-\eta)}\Phi(\eta)\,d\eta\right\}, \tag{11}$$

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1) This reads: *$\{\mathfrak{C}_r\}$ is associated to the interval* $(\delta, -\delta)$. As we have defined in [T], Chapter I, § 3, Definition III, this means that $\{\mathfrak{C}_r\}$ satisfies the following two conditions:

Condition 1°. The distances d_r between the contour $\mathfrak{C}_r$ and the origin tends to infinity as $r\to\infty$.

Condition 2°. As r tends to infinity we have

$$\int_{\mathfrak{C}_r^{(+)}}|e^{-\lambda\delta}|\,|d\lambda|=O(1); \quad \int_{\mathfrak{C}_r^{(-)}}|e^{\lambda\delta}|\,|d\lambda|=O(1),$$

where $\mathfrak{C}_r^{(+)}$ resp. $\mathfrak{C}_r^{(-)}$ denote the parts of $\mathfrak{C}_r$ located in the positive resp. the negative half-planes of λ-plane.

Since we have already assumed Condition 1°, Condition 2° is essential.

2) This reads: *$G(\lambda)$ is O-associated to $\{\mathfrak{C}_r\}$ with the index 0 in the interval* $(\delta, b-\delta)$. The meaning is as follows: $G(\lambda)$ has no zero-points on any contour of the sequence (this has been already implicitly assummed) and further

$$\int_{\mathfrak{C}_r^{(\pm)}}\left|\frac{e^{\lambda q}}{G(\lambda)}\right|\,|d\lambda|=O(1) \qquad (\text{as } r\to\infty)$$

uniformly concerning q in $\delta\leqq q\leqq b-\delta$.

3) Incidently we have the opportunity to remark that in the hypothesis to Theorem XII. I in [T] p. 286 we must add the condition that $f(x)$ is a solution of the function equation $\Lambda f(x)=0$ in $-\infty<x<\infty$. The employment of the Fejér-sum of Cauchy series and Condition 3° are then avoidable. *Thus* Theorem XII. I *will become to possess no existence-value,* and it should be replaced by the present Theorem II.

where l is the linear functional discussed in §§ 1–2 and $\Phi(\eta)$ is a certain function belonging to $L^q(0, 2\pi)$ with $q > 1$.

Assumption II. For a certain given a, the linear functional l^a is defined by

$$(12) \qquad l^a_\xi\{f(\xi)\} \equiv l_\xi\{e^{a\xi}f(\xi)\} \equiv \int_0^b e^{a\xi}f(\xi)\,d\varphi(\xi)\,,$$

and there is a sequence of contours $\{\mathfrak{C}_r\}(r=1, 2, \ldots\ldots)$ such that, for each fixed z in the finite complex z-plane, the Cauchy series of e^{zt} (as functions of t) with respect to the linear functional l^a, which we denote by $\{S^a_{\mathfrak{C}_r}(x, e^{zt})\}$ $(r=1, 2, \ldots\ldots)$, possesses the property:

$$(13) \qquad \lim_{r\to\infty}\int_0^b |e^{zt} - S^a_{\mathfrak{C}_r}(t;\, e^{zt})|^p dt = 0\,,$$

with p subjected to the condition $1/p+1/q=1$.

Theorem III. Under the Assumptions I–II, we have, for each fixed z,

$$(14) \qquad C(z) = \lim_{r\to\infty}\frac{G(z+a)}{2\pi i}\oint_{\mathfrak{C}_r}\frac{C(\lambda)}{G(\lambda+a)(z-\lambda)}\,d\lambda\,.$$

Proof: This can be proved quite similarly as in Theorem I, in view of the facts that the generating function of l^a is $G(\lambda+\sigma)$ and that

$$(15) \qquad S^a_{\mathfrak{C}}(t;\, e^{zt}) \equiv \frac{1}{2\pi i}\oint_{\mathfrak{C}}\frac{e^{t\lambda}l^a_\xi\left\{e^{\lambda\xi}\int_0^\xi e^{z\eta}e^{-\lambda\eta}\,d\eta\right\}}{G(\lambda+a)}\,d\lambda$$
$$= \frac{G(z+a)}{2\pi i}\oint_{\mathfrak{C}}\frac{e^{t\lambda}d\lambda}{G(\lambda+a)(z-\lambda)}\,.$$

In case when the Assumption II is valid not only for a certain single value of a but also each value belonging to certain range in the complex plane by respective suitable choice of the sequence of contours, the relation[1] may be recognised as the consistency property of the series of the right-hand side of (15). The consistency of the cardinal series follows from the facts that in this case $G(\lambda) = \sin \lambda\pi$[2] and consequently that the associated Cauchy series, which coincides particulary

1) Their results and related literatures are gathered together in the treatise of *J. M. Whittaker*: Interpolatory Function Theory. Cambridge Tracts, **33** (1935).

2) The cardinal series can be considered as a special case of the series (15) by the following conventions; we define

$$\varphi(\xi) = \begin{cases} -\dfrac{1}{2i} & [\xi=-\pi]\,. \\ 0 & [-\pi<\xi<\pi]\,. \\ \dfrac{1}{2i} & [\xi=\pi]\,. \end{cases}$$

And we consider functions defined on the imaginary axis: the linear functional l is defined by

$$l_\xi\{f(i\xi)\} \equiv \int_{-\pi}^{\pi} f(i\xi)\,d\varphi(\xi) = \{f(i\pi)-f(-i\pi)\}/2i\,,$$

the generating function is $l_\xi\{e^{\lambda\xi}\} = \sin\lambda$.

with the ordinary Fourier series, satisfies the Assumption II for any finite complex number a.[1]

The author wish to remark that the problem that, given a sequence of complex numbers $\{a_{n,k}\}$ ($n=1, 2, 3, \ldots\ldots; k=1, 2, \ldots\ldots, m_n$), where each m_n[2] is the order of multiplicy of each λ_n as zero-point of $G(\lambda)$, what condition will imply that there is a function $\Phi(\eta)$ belonging to $L^q(0, b)$ such that

$$\lim_{r\to\infty}\sum_{n=0}^{N_r}\sum_{k=1}^{m_n}\frac{a_{n,k}}{(z-\lambda_n)^k}=l_\xi\left\{\int_0^\xi e^{z(\xi-\eta)}\Phi(\eta)\,d\eta\right\} \tag{16}$$

is remaining for further investigation. The form $l_\xi\left\{\int_0^\xi e^{z(\xi-\eta)}d\Phi(\eta)\right\}$, where $\Phi(\eta)$ is a continuous function, will deserve a consideration in connection with the result of J. M. Whittaker[3] on the cardinal series.

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1) In his treatise loc. cit., (p. 71) J. M. Whittaker said "The cardinal series is particularly favoured because the integral function $H(Z)$ ($G(\lambda)$ in our notation) associated with it happens to be $\sin \pi Z$ ($\sin \pi\lambda$ in our notation)." And he emphasised these pecurialities in two points: the first is the orthogonality, i. e.

$$\int_0^1 \sin \pi nt \sin \pi mt \, dt=\delta_{n,m}$$

and the second is the addition theorem. But so far as Theorem III concerns, it is to be noted that the asymptotic behaviour of $G(\lambda)$ plays the most important rôle, as may be seen from the Assumptions I–II.

2) The multiplicity of $\{\lambda_n\}$ is the positive integer m_n such that

$$G^{(s)}(\lambda_n)=0 \qquad (s=0, 1, 2, \ldots\ldots, m_n-1)$$
$$G^{(m_n)}(\lambda_n)\neq 0\,.$$

3) J. M. Whittaker: loc. cit., pp. 67–71. Cf. also his original paper: The "Fourier" theory of the cardinal function. Proc. Edinburgh Math. Soc. **1** (1929), 169–176. Our Theorem III concerns itself more intimately with the results of W. L. Ferrar. Cf. *Ferrar*: Proc. Royal Soc. Edinburgh. **45** (1925); **46** (1925); **47** (1927).