# 61. *Two Fixed-point Theorems Concerning Bicompact Convex Sets.*

By Shizuo KAKUTANI.

Mathematical Institute, Osaka Imperial University.

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**1.** In this paper we are concerned with two kinds of fixed-point theorems, which are not essentially new. Theorem 1 is already given by A. Markov[1] and Theorem 2 is a generalisation of a result of J. v. Neumann[2] and W. Maak.[3] But the simple way of proving Theorem 1 and the general formulation of Theorem 2 may be regarded with some interest. It is also to be noticed, that, in spite of their similar appearance, these theorems are treated in entirely different ways. Moreover, we shall give some applications of these theorems, which illustrate the importance of these fixed-point theorems. Only the results and the brief summary of their proofs are given, the details being left to a subsequent paper, which will be published elsewhere.

**2.** Let $B$ be a convex set in some linear space $E$, and $\Gamma$ a family of transformations $\varphi(x)$ of $B$ into itself. $\varphi(x)$ is called to be *affine* if for any $x, y \in B$ and $\lambda, \mu \geqq 0$, $\lambda+\mu=1$, we have $\varphi(\lambda x+\mu y)=\lambda\varphi(x)+\mu\varphi(y)$; and $\Gamma$ is called to be *abelian* if for any $\varphi, \psi \in \Gamma$ and $x \in B$ we have $\varphi\big(\psi(x)\big)=\psi\big(\varphi(x)\big)$.

*Theorem 1. Let $B$ be a non-vacuous, convex, bicompact subset of a locally convex linear topological space $E$, and let $\Gamma$ be an abelian family of continuous affine transformations $\varphi(x)$ of $B$ into itself; then there is a point $x \in B$ such that we have $\varphi(x)=x$ for any $\varphi \in \Gamma$.*

A. Markov's original proof uses the fixed-point theorem of A. Tychonoff.[4] Since this theorem is a generalisation of Brouwer's fixed-point theorem which is valid for general (not necessarily affine!) continuous transformations, a direct proof will be desirable.

To prove Theorem 1 we proceed as follows: Consider the totality $\Gamma^*$ of all the transformations $\varphi^*(x)$ of the form:

$$\varphi^*(x)=\frac{1}{n}\Big(x+\varphi(x)+\cdots+\varphi^{n-1}(x)\Big),$$

$n=1, 2, \dots$; $\varphi \in \Gamma$, where $\varphi^k(x)$ denotes the $k$-th iterate of $\varphi(x)$. $\varphi^*(x)$ is also a continuous affine transformation of $B$ into itself and $\Gamma^*$ is abelian. It will then be easy to see that $B_1 \equiv \prod_{\varphi^* \in \Gamma^*} \varphi^*(B)$ is not empty and that any point $x \in B_1$ is a desired fixed-point.

1) A. Markov: Quelques théorèmes sur les ensembles abéliens, C. R. URSS, **2** (1936), p. 311.

2) J. v. Neumann: Almost periodic function in groups I, Trans. Amer. Math. Soc., **36** (1934), p. 445.

3) W. Maak: Eine neue Definition der fastperiodischen Funktionen, Abh. math. Semin. Hansisch. Univ., **11** (1936), p. 240.

4) A. Tychonoff: Ein Fixpunktsatz, Math. Ann., **111** (1935), p. 767.

*Corollary. Let $B$ be the same as in Theorem 1 and let $\Gamma$ be a soluble group of continuous affine transformations $\varphi(x)$, which map $B$ univalently on itself; then there is a point $x \in B$ such that we have $\varphi(x)=x$ for any $\varphi \in \Gamma$.*

**3.** *Application of Theorem 1.* As an application of Theorem 1 we shall show here that the theorem of Banach[1] on the extension of linear functionals may be obtained from Theorem 1. The existence of such a relation may be conjectured from the fact that we can treat the problem of measure (invariant under abelian transformation groups) starting from any one of these theorems.

*Banach's Theorem. Let $E$ be a linear space and $f(x)$ a linear functional defined over some linear subspace $E_0$ of $E$. If there is a functional $p(x)$ defined over $E$ such that*

$$p(tx)=tp(x),$$

$$p(x+y) \leqq p(x)+p(y)$$

*for any $x$, $y \in E$, $t \geqq 0$, and*

$$f(x) \leqq p(x)$$

*for any $x \in E_0$, then there exists a linear functional $F(x)$ defined over $E$ which satisfies*

$$F(x) \leqq p(x)$$

*for any $x \in E$ and*

$$F(x)=f(x)$$

*for any $x \in E_0$.*

In order to prove this, we consider as $B$ the set of all the functionals $F(x)$ (not necessarily linear!) defined over $E$ such that

$$-p(-x) \leqq F(x+y)-F(y) \leqq p(x)$$

for any $x$, $y \in E$ and

$$F(x)=f(x)$$

for any $x \in E_0$; and as $\Gamma$ the group of all linear transformations generated by the linear transformations of the following two types:

$$S_t\{F(x)\}=\frac{F(tx)}{t}, \qquad t \gtrless 0,$$

$$T_y\{F(x)\}=F(x+y)-F(y), \qquad y \in E.$$

That all the conditions of Corollary of Theorem 1 are satisfied for these $B$ and $\Gamma$ may be verified,[3] and the fixed-point thus obtained is the required linear extension. Indeed, $S_t\{F(x)\}=F(x)$ for any $t \gtrless 0$

1) S. Banach: Théorie des opérations linéaires, Warsaw, 1932, p. 27.

2) S. Banach, loc. cit., (5), p. 30–34 and A. Markov, loc. cit., (1).

3) $B$ is bicompact in A. Tychonoff's topology. Compare:
A. Tychonoff: Über einen Funktionenraum, Math. Ann., **111** (1935), p. 762.
Only that $B$ is not empty is difficult to prove.

means that $F(x)$ is homogeneous and $T_y\{F(x)\}=F(x)$ for any $y\in E$ means that $F(x)$ is additive.

As the second application we shall remark that the result of R. P. Agnew and A. P. Morse[1] is also a direct consequence of Theorem 1.

**4.** Let $E$ be a locally convex linear topological space. A continuous affine transformation $\varphi(x)$ of $E$ on itself is called to be *congruent*, if there is a neighbourhood system $\mathfrak{U}=\{U\}$ (of a zero element) of $E$ such that we have $\varphi(x+U)=\varphi(x)+U$ for any $x\in E$ and $U\in\mathfrak{U}$. We may assume that every $U$ is convex. In the case of a normed linear space, an isometric transformation is congruent.

*Theorem 2. Let $B$ be a non-vacuous, convex, bicompact subset of a locally convex linear topological space $E$, and let $\Gamma$ be a group of affine congruent transformations $\varphi(x)$ of $E$ on itself, which map $B$ univalently on itself, then there is a point $x\in B$ such that we have $\varphi(x)=x$ for any $\varphi\in\Gamma$.*

To show the relation between Theorem 2 and the result of J. v. Neumann and W. Maak, let $G$ be any group and $f(x)$ a real-valued almost periodic function defined on $G$. If we consider as $E$ the linear space of all bounded real-valued functions $g(x)$ defined on $G$ with the norm $\|g\|=\underset{x\in G}{\text{l. u. b.}}\ |g(x)|$, then $E$ is a normed linear space and the aggregate $A$ of all the functions of the form: $f_a(x)=f(xa)$, $a\in G$, is a totally bounded set in $E$. Let $B$ be the least convex closed set containing $A$. $B$ is compact and is the set of all the functions $g(x)$ defined over $G$ which can be uniformly approximated by the functions of the form:

$$f^*(x)\equiv\lambda_1 f(xa_1)+\lambda_2 f(xa_2)+\cdots+\lambda_n f(xa_n),$$
$$a_1, a_2, \ldots, a_n\in G;\ \lambda_1, \lambda_2, \ldots, \lambda_n\geqq 0,\ \lambda_1+\lambda_2+\cdots+\lambda_n=1;\ n=1, 2, \ldots.$$

If we define for any $g\in E$ $T_a\{g(x)\}=g(xa)$, $a\in G$, then $T_a$ is an affine isometric transformation of $E$ on itself, which maps $B$ univalently on itself, and the existence of a fixed point, on which we insist in Theorem 2, is nothing but the existence of a mean of the almost periodic function $f(x)$.

The proof of Theorem 2 relies on Maak's method. The combinatorial lemma, which was introduced by him successfully, plays here also a fundamental rôle. It is to be remarked, that Theorem 2 holds in any locally convex linear topological space, while both J. v. Neumann[2] and W. Maak[3] require a kind of countability axioms on $E$. The proof proceeds similarly in this general case. The uniqueness of the mean, however, does not follow in any case, since the assumption in Theorem 2 is much weaker than in the case of almost periodic functions.

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1) R. P. Agnew and A. P. Morse: Extensions of linear functionals, with applications to limits, integrals, measures and densities, Annals of Math., **39** (1938), p. 20.

2) S. Bochner and J. v. Neumann: Almost periodic function in groups II, Trans. Amer. Math. Soc., **37** (1935), p. 21.

J. v. Neumann: On complete topological spaces, Trans. Amer. Math. Soc., **37** (1935), p. 1.

3) W. Maak: Abstrakte fastperiodische Funktionen, Abh. math. Semin. Hansisch. Univ., **11** (1936), p. 367.

**5.** We now introduce a uniform topology in $\Gamma$: For any $\varphi_0 \in \Gamma$ we define its neighbourhood $\mathfrak{V}(\varphi_0, U)$ as the totality of all $\varphi \in \Gamma$ for which $\varphi(x) - \varphi_0(x) \in U$ for any $x \in B$, $U$ denoting any neighbourhood of zero element of $E$. With this neighbourhood system the total boundedness of $\Gamma$ may be defined as usual.

*Corollary of Theorem 2. Let $B$ be a non-vacuous, convex, bicompact subset of a locally convex linear topological space $E$, and let $\Gamma$ be a totally bounded group of continuous affine transformations $\varphi(x)$ of $E$ on itself, which map $B$ univalently on itself; then there is a point $x \in B$ such that we have $\varphi(x) = x$ for any $\varphi \in \Gamma$.*