44. *Weak Topology and Regularity of Banach Spaces.*

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1. *Introduction.* Let E be a Banach space (element x with norm $\|x\|$), and let $\bar{E}$ be its conjugate space (i. e. the space of all bounded linear functionals $f(x)$ defined on E, with norm $\|f\| = \underset{\|x\| \leq 1}{\text{l. u. b.}} |f(x)|$). $\bar{E}$ is also a Banach space and its conjugate space $\bar{\bar{E}}$ may be considered. E is called to be *regular* if we have $E = \bar{\bar{E}}$, or equivalently, if every bounded linear functional $X_0(f)$ defined on $\bar{E}$ may be represented in the form: $X_0(f) = f(x_0)$ for any $f \in \bar{E}$, where x_0 is a point from E.

We shall give, in the first part of this paper (§§ 2 and 3), some conditions for the regularity of E. This problem was investigated by several authors and many interesting results were obtained. Our principal idea is to use the *weak topologies* in E and in $\bar{E}$. It seems to me that too little attention has been paid to the weak topologies in Banach spaces, while on the contrary in the theory of Hilbert space weak topology plays an essential rôle. It will be shown in this paper how weak topologies are successfully introduced into such problems. We shall state only the definition of weak topologies and a few fundamental theorems, the rest being left to another occasion.

In the second part of this paper (§ 5), we shall prove that every uniformly convex Banach space is regular (Theorem 3).[1] From this follows easily that Mean Ergodic Theorem holds true in uniformly convex Banach spaces. A direct proof of this fact was obtained recently by Garrett Birkhoff.[2]

The proof of Theorem 3 relies on Helly's theorem.[3] Mr. Yukio Mimura has kindly informed me of a simple and elegant proof of this theorem. This proof is given in § 4. I express my hearty thanks to Mr. Yukio Mimura.

2. *Weak topologies.* (1) *Weak topology in E.* For any $x_0 \in E$, its weak neighbourhood $U_1(x_0, f_1, f_2, \dots, f_n, \varepsilon)$ is defined as the totality of all the points $x \in E$ such that $|f_i(x) - f_i(x_0)| < \varepsilon$ for $i = 1, 2, \dots, n$, where $\{f_i(x)\}$ $(i = 1, 2, \dots, n)$ is an arbitrary system of bounded linear functionals defined on E and $\varepsilon > 0$ is an arbitrary positive number.

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1) This theorem was recently proved by D. Milman in a different way. D. Milman: On some criteria for the regularity of spaces of type (*B*), C. R. URSS, **20** (1938), 243–246. D. Milman's proof uses the notion of transfinite closedness. It is our purpose to avoid, as far as possible, the use of transfinite method in the theory of Banach spaces.

2) G. Birkhoff: The mean ergodic theorem, Duke Math. Journ., **5** (1939), 19–20.

3) E. Helly: Über Systeme linearer Gleichungen mit unendlich vielen Unbekannten, Monatsh. für Math. und Phys., **31** (1921), 60–91.

The topology (WT_1) of E defined by this neighbourhood system is called the *weak topology of E as elements.*

(2) *Weak topologies in $\bar{E}$.* For any $f_0 \in \bar{E}$ its weak neighbourhood $U_2(f_0, x_1, x_2, \dots, x_n, \varepsilon)$ is defined as the totality of all the bounded linear functionals $f(x)$ defined on E such that $|f(x_i)-f_0(x_i)| < \varepsilon$ for $i=1, 2, \dots, n$, where $\{x_i\}$ $(i=1, 2, \dots, n)$ is an arbitrary system of points from E and $\varepsilon > 0$ is an arbitrary positive number. The topology (WT_2) of $\bar{E}$ defined by this neighbourhood system is called the *weak topology of $\bar{E}$ as functionals.*

It is clear that we can also introduce in $\bar{E}$ a weak topology as elements (considering $\bar{E}$ itself as a Banach space). The *weak topology* (WT_3) *of E as elements* is indeed defined by the following neighbourhood system: for any $f_0 \in \bar{E}$ its weak neighbourhood $U_3(f_0, X_1, X_2, \dots, X_n, \varepsilon)$ is the set of all the elements f of $\bar{E}$ such that $|X_i(f)-X_i(f_0)| < \varepsilon$ for $i=1, 2, \dots, n$, where $\{X_i(f)\}$ $(i=1, 2, \dots, n)$ is an arbitrary system of bounded linear functionals defined on $\bar{E}$ and $\varepsilon > 0$ is an arbitrary positive number.

Since, for any fixed $x_0 \in E$, $f(x_0)$ may be considered as a bounded linear functional defined on $\bar{E}$, every neighbourhood in the topology (WT_2) is also a neighbourhood in the topology (WT_3). Hence the topology (WT_3) is not weaker than the topology (WT_2). The converse is not always true and we have the

Theorem 1. The necessary and sufficient condition that a Banach space E is regular, is that the two weak topologies (WT_2) *and* (WT_3) *are equivalent in $\bar{E}$.*

This theorem may be considered as a generalization of a theorem of S. Banach.[1] For the proof of Theorem 1 we need the Theorem 2 in § 3. We omit the proof.

3. *Weakly closed and regularly closed set of functionals.* Let Γ be a set of bounded linear functionals defined on E (i.e. Γ is a subset of $\bar{E}$). Γ is called to be weakly closed (as a set of functionals), if it is closed in the weak topology (WT_2) in $\bar{E}$, and Γ is called to be regularly closed (as a set of functionals), if there exists for any $f_0 \in \bar{E}-\Gamma$ a point $x_0 \in E$ such that $f_0(x_0) \neq 0$ and $f(x_0)=0$ for any $f \in \Gamma$. It is clear that the regular closedness implies the weak closedness, and as to the converse we have the

Theorem 2. If Γ is a linear subset of $\bar{E}$, then the weak closedness of Γ is equivalent to the regular one.

The proof of Theorem 2 may be easily carried out in an elementary way. We omit the proof.

Corollary 1. For a linear subset of E, the three notions of closure: weak, regular and the transfinite are equivalent.

The coincidence of the last two closure properties was proved by S. Banach.[2]

1) S. Banach: Théorie des opérations linéaires, p. 131, Théorème 8.

2) S. Banach, loc. cit., p. 121, lemme 3.

Corollary 2. In order that a Banach space E is regular, it is necessary and sufficient that E is closed in $\bar{\bar{E}}$ in the weak topology of $\bar{\bar{E}}$ as functionals defined on $\bar{E}$.

Since E is total as a set of bounded linear functionals defined on $\bar{E}$, the necessary and sufficient condition for $E=\bar{\bar{E}}$ is that E is regularly closed in $\bar{\bar{E}}$ as a set of functionals defined on $\bar{E}$. This is a result of S. Banach,[1] and our corollary is a direct consequence of this fact and Theorem 2.

This corollary is also a direct consequence of Helly's theorem, which will be proved in § 4. H. Goldstine[2] also obtained the analogous result, that the necessary and sufficient condition that E is regular is that E is δ-weakly closed in $\bar{\bar{E}}$. It will be easily seen that the result of H. Goldstine is a direct consequence of Helly's theorem, and that in this case the δ-weak closedness of E is equivalent to the weak closedness of E in the weak topology of $\bar{\bar{E}}$ as functionals.

4. *Helly's theorem. Let E be a Banach space and let $\{f_i(x)\}$ $(i=1, 2, \dots, n)$ be a system of bounded linear functionals defined on E. Given a system of real numbers $\{c_i\}$ $(i=1, 2, \dots, n)$ and a positive number $M>0$, the necessary and sufficient condition that there exists for any $\varepsilon>0$ a point $x_0 \in E$ such that*

$$\|x_0\| < M+\varepsilon \quad \text{and} \quad f_i(x_0)=c_i \quad \text{for} \quad i=1, 2, \dots, n, \tag{1}$$

is that we have

$$\left\| \sum_{i=1}^{n} \lambda_i c_i \right\| \leqq M \cdot \left\| \sum_{i=1}^{n} \lambda_i f_i \right\| \tag{2}$$

for any system of real numbers $\{\lambda_i\}$ $(i=1, 2, \dots, n)$.

Proof of Helly's theorem due to Y. Mimura. Since the necessity of the condition is clear, we shall only prove that the condition is sufficient. Consider the linear transformation $x \to Tx = \{f_1(x), f_2(x), \dots, f_n(x)\}$, which maps the Banach space E into the n-dimensional Euclidean space R^n. Let S be the sphere $\|x\| < M+\varepsilon$ of E. We shall prove that the image $T(S)$ of S in R^n by this transformation T contains the point $P_0 = \{c_1, c_2, \dots, c_n\}$ of R^n. For this purpose, assume on the contrary that P_0 does not belong to $T(S)$. Since $T(S)$ is clearly convex, there exists in R^n an $(n-1)$-dimensional hyperplane which passes through P_0 and which touches the convex set $T(S)$. In other words, there exists a system of real numbers $\{\lambda_i\}$ $(i=1, 2, \dots, n)$ such that $\sum_{i=1}^{n} \lambda_i c_i = 1$ and $\sum_{i=1}^{n} \lambda_i \xi_i \leqq 1$ for any point $\{\xi_1, \xi_2, \dots, \xi_n\} \in T(S)$, or equivalently $\sum_{i=1}^{n} \lambda_i f_i(x) \leqq 1$ for any $x \in E$ with $\|x\| < M+\varepsilon$. Since the least upper bound of $\sum_{i=1}^{n} \lambda_i f_i(x)$ for $\|x\| < M+\varepsilon$ is $(M+\varepsilon) \cdot \left\| \sum_{i=1}^{n} \lambda_i f_i \right\|$, we have

1) S. Banach, loc. cit., p. 117, Remarque.

2) H. Goldstine: Weakly complete Banach spaces, Duke Math. Journ., **4** (1938), 125–131.

$(M+\varepsilon)\cdot\left\|\sum_{i=1}^{n}\lambda_i f_i\right\|\leqq 1=\sum_{i=1}^{n}\lambda_i c_i$. Since $\varepsilon>0$ is positive, this contradicts the assumption (2). Hence the point $P_0=\{c_1, c_2, \dots, c_n\}$ must belong to $T(S)$, i. e., there exists a point $x_0 \in E$ such that the condition (1) is satisfied.

Corollary. Let $X_0(f)$ be an arbitrary bounded linear functional defined on the conjugate space $\bar{E}$ of E. Then there exists, for any system of bounded linear functionals $\{f_i(x)\}$ $(i=1, 2, \dots, n)$ defined on E, and for any positive number $\varepsilon>0$, a point $x_0 \in E$ such that

(3) $\|x_0\|<\|X_0\|+\varepsilon$ *and* $X_0(f_i)=f_i(x_0)$ *for* $i=1, 2, \dots, n$.

5. *Mean Ergodic Theorem in uniformly convex Banach spaces.* A Banach space is called to be *uniformly convex* if there exists for any $\varepsilon>0$ a $\delta(\varepsilon)>0$ such that $\|x\|\leqq 1$, $\|y\|\leqq 1$ and $\|x-y\|\geqq\varepsilon$ imply $\|x+y\|\leqq 2\,\big(1-\delta(\varepsilon)\big)$. It is clear that we may assume that $\delta(\varepsilon)$ is continuous in ε.

Theorem 3. Every uniformly convex Banach space is regular.

Proof. Let E be a uniformly convex Banach space. We shall prove that, for any bounded linear functional $X_0(f)$ defined on the conjugate space $\bar{E}$ of E, there exists a point $x_0 \in E$ such that $X_0(f)=f(x_0)$ for any $f\in\bar{E}$. Since the case $X_0(f)\equiv 0$ is trivial, we may assume that $\|X_0\|=\underset{|f|\leq 1}{\text{l. u. b.}}\ |X_0(f)|=1$. Then there exists a sequence $\{f_i\}$ $(i=1, 2, \dots)$ from $\bar{E}$ such that $\|f_i\|=1$ and $X_0(f_n)>1-1/n$ for $n=1, 2, \dots$. Taking in consideration only the first n terms $f_1, f_2, \cdots, f_n$ of this sequence, we see, from the corollary of Helly's theorem, that there exists for each n a point $x_n \in E$ such that $\|x_n\|<1+1/n$ and $f_i(x_n)=X_0(f_i)$ for $i=1, 2, \dots, n$.

We shall prove that the sequence $\{x_n\}$ $(n=1, 2, \dots)$ thus obtained is a fundamental sequence in E. For this purpose, assume on the contrary that this is not the case. Then there exists a positive number $\varepsilon>0$ and an increasing sequence of positive integers $n_1<m_1<n_2<m_2<\cdots<n_k<m_k<\cdots$ such that $\|x_{n_k}-x_{m_k}\|\geqq\varepsilon$ for $k=1, 2, \dots$. Since $\overline{\lim_{n\to\infty}}\|x_n\|\leqq 1$, we have, from the uniform convexity of E, $\overline{\lim_{k\to\infty}}\|x_{n_k}+x_{m_k}\|\leqq 2\,\big(1-\delta(\varepsilon)\big)<2$.

On the other hand, from the definition of x_{n_k} and x_{m_k} $(n_k<m_k)$, we have $f_{n_k}(x_{m_k})=X_0(f_{n_k})$ and $f_{n_k}(x_{n_k})=X_0(f_{n_k})$ for $k=1, 2, \dots$. This will, however, lead to the contradiction with the inequality just obtained; for, we have

$$2(1-1/n_k)<2\cdot X_0(f_{n_k})=f_{n_k}(x_{n_k}+x_{m_k})\leqq\|f_{n_k}\|\cdot\|x_{n_k}+x_{m_k}\|=\|x_{n_k}+x_{m_k}\|$$

for $k=1, 2, \dots$, and consequently $\underline{\lim_{k\to\infty}}\|x_{n_k}+x_{m_k}\|\geqq 2\cdot\lim_{k\to\infty}(1-1/n_k)=2$.

Thus we have proved that the sequence $\{x_n\}$ $(n=1, 2, \dots)$ is a fundamental sequence in E. Let x_0 be the limiting point of this sequence. This x_0 clearly satisfies the condition:

(4) $\|x_0\|=1$ and $f_n(x_0)=X_0(f_n)$ for $n=1, 2, \dots$.

The second relation is clear, and the first follows from the fact that we have $1-1/n < X_0(f_n)=f_n(x_0) \leqq \|f_n\| \cdot \|x_0\| = \|x_0\|$ for $n=1, 2, \dots$.

Now we shall prove that this x_0 is a required one. For this purpose, we first remark that such an x_0 is uniquely determined by the relation (4). This follows directly from the uniform convexity of E. For, if there exists another x_0' with the same property (4), then we have $\|x_0+x_0'\| \leqq 2\left(1-\delta(\|x_0-x_0'\|)\right) < 2$ and $f_n(x_0+x_0')=2\cdot X_0(f_n)$ for $n=1, 2, \dots$. This is, however, a contradiction, since the latter implies $2(1-1/n) < 2\cdot X_0(f_n)=f_n(x_0+x_0') \leqq \|f_n\|\cdot\|x_0+x_0'\| = \|x_0+x_0'\|$ for $n=1, 2, \dots$.

Next we shall prove that this x_0 satisfies $X_0(f)=f(x_0)$ for any $f\in\bar{E}$. In order to prove this, take an arbitrary $f_0\in\bar{E}$ and consider the sequence $\{f_n\}$ $(n=0, 1, 2, \dots)$. If we define $x_n'\in E$ by the conditions: $\|x_n'\| < 1+1/n$ and $X_0(f_i)=f_i(x_n')$ for $i=0, 1, 2, \dots, n$, (using the corollary of Helly's theorem), then the sequence $\{x_n'\}$ $(n=1, 2, \dots, n)$ thus obtained is also a fundamental sequence in E. This may be proved in just the same way as in the preceding. Let x_0' be the limiting point of this sequence $\{x_n'\}$ $(n=1, 2, \dots)$. x_0' also satisfies (4) and the additional relation: $f_0(x_0')=X_0(f_0)$. Since x_0 is uniquely determined by (4), we must have $x_0=x_0'$, and consequently x_0 satisfies the relation $f_0(x_0)=X_0(f_0)$. Since f_0 is an arbitrary functional from $\bar{E}$, the proof of Theorem 3 is hereby completed.

Since every regular Banach space is locally weakly compact we have the

Corollary. Every uniformly convex Banach space is locally weakly compact.

Using the last corollary we can easily prove the following

Theorem 4. (Mean Ergodic Theorem in regular or uniformly convex Banach spaces). Let T be a bounded linear transformation which maps a regular or a uniformly convex Banach space E into itself. If there exists a constant C such that $\|T^n\| \leqq C$ for $n=1, 2, \dots$, then there exists a bounded linear transformation T_1, which maps E into itself, such that

$$\lim_{n\to\infty} \frac{1}{n}(T+T^2+\cdots+T^n)x = T_1 x$$

exists strongly for any $x\in E$, and $TT_1=T_1T=T_1^2=T_1$.

This is indeed a direct consequence of the Mean Ergodic Theorem in Banach spaces, which was proved by K. Yosida[1] and the author.[2] This result was recently proved directly by Garrett Birkhoff for uniformly convex Banach spaces under the assumption that $C=1$.

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1) K. Yosida: Mean ergodic theorem in Banach spaces, Proc. **14** (1938), 292–294.

2) S. Kakutani: Iteration of linear operations in complex Banach spaces, Proc. **14** (1938), 295–300.