66. Some Results in the Operator-Theoretical Treatment of the Markoff Process.

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Let us denote by $P(t, E)$ the transition probability that a point $t \in \Omega$ is transferred, by a simple Markoff process, into a Borel set E of Ω after the elapse of a unit-time. We have always $P(t, E) \geqq 0$ and $P(t, \Omega)=1$. We shall assume that $P(t, E)$ is completely additive for Borel sets E if t is fixed, and that $P(t, E)$ is Borel measurable in t if E is fixed. Then the probability $P^{(n)}(t, E)$ that t is transferred into E after the elapse of n unit-times is given recurrently by

$$P^{(n)}(t, E)=\int_{\Omega} P^{(n-1)}(t, ds)P(s, E), \ n=2, 3, \dots ; \ P^{(1)}(t, E)=P(t, E),$$

where the integration is of Radon-Stieltjes type.

Consider the complex Banach space $(\mathfrak{M})$ of all the complex-valued completely additive set functions $x(E)$ defined for all Borel sets E of Ω. For any $x(E) \in (\mathfrak{M})$, its norm is defined by $\|x\|=$ total variation of $|x(E)|$ on Ω. Then the integral operator

$$x \rightarrow T(x)=y: \quad y(E)=\int_{\Omega} x(dt)P(t, E)$$

is a bounded linear operation which maps $(\mathfrak{M})$ into itself and $\|T\|=1$. On the other hand, if we consider the complex Banach space (M^*) of all the complex-valued bounded measurable functions $x(t)$ defined on Ω, with $\|x\|=\underset{t \in \Omega}{\text{l. u. b.}} |x(t)|$ as its norm, then

$$x \rightarrow \bar{T}(x)=y: \quad y(t)=\int_{\Omega} P(t, ds)x(s)$$

is also a bounded linear operation which maps (M^*) into itself and $\|\bar{T}\|=1$.

Our main object is to investigate the asymptotic behaviour of $P^{(n)}(t, E)$ for large n. Since, as is easily seen, T^n and $\bar{T}^n$ are the integral operators defined by the kernel $P^{(n)}(t, E)$, our problem is transformed into the investigation of the iterations T^n and $\bar{T}^n$ of T and $\bar{T}$ respectively.

This investigation was carried out by K. Yosida.[1] Under the condition of N. Kryloff-N. Bogolioùboff:[2]

1) K. Yosida: Operator-theoretical treatment of the Markoff process, Proc. **14** (1938), 363–367.

2) N. Kryloff-N. Bogolioùboff: Sur les probabilités en chaîne, C. R. Paris, **204** (1937), 1386–1388.

N. Kryloff-N. Bogolioùboff: Les propriétés ergodiques des suites des probabilités en chaîne, C. R. Paris, **204** (1937), 1454–1455.

(K) $\begin{cases}\text{there exist an integer } m \text{ and a completely continuous linear ope-} \\ \text{ration } V, \text{ which maps } (\mathfrak{M}) \text{ into itself, such that } \|T^m - V\| < 1,\end{cases}$

he has obtained the results of M. Fréchet,[1] J. L. Doob[2] and W. Doeblin[3] in a more precise form. In the present paper, we shall develop this idea of K. Yosida, and shall obtain some more precise results. Our principal idea is to use both the spaces $(\mathfrak{M})$ and (M^*). Although the two operations T and $\bar{T}$ are not conjugate to each other, these play, in essential, the same rôle.

Lemma 1 (*Uniform Ergodic Theorem*).[4] *Under the condition* (K), *the proper values* λ *of* T *of modulus* 1 *are finite in number, and if we denote these by* $\lambda_1, \lambda_2, \dots, \lambda_k$, *then* T^n *is decomposed into the form:*

$$T^n = \sum_{i=1}^{k} \lambda_i^n T_{\lambda_i} + S^n, \quad n = 1, 2, \dots,$$

where $\{T_{\lambda_i}\}$ $(i = 1, 2, \dots, k)$ *is a completely continuous linear operator such that* $\|T_{\lambda_i}\| = 1$, $T T_{\lambda_i} = T_{\lambda_i} T = \lambda_i T_{\lambda_i}$, $T_{\lambda_i}^2 = T_{\lambda_i}$, $T_{\lambda_i} T_{\lambda_j} = 0$ $(i \neq j)$, $T_{\lambda_i} S = S\, T_{\lambda_i} = 0$ $(i = 1, 2, \dots, k)$ *and* $\|S^n\| \leqq \dfrac{M}{(1+\varepsilon)^n}$, $n = 1, 2, \dots$, *with positive constants* M *and* ε.

As a corollary to this, we have

Lemma 2. Under the condition (K), $\lambda = 1$ *is a proper value of* T *and there exists a kernel* $P_1(t, E)$ *with* $P_1(t, E) \geqq 0$, $P(t, \Omega) = 1$ *such that*

$$\int_\Omega P_1(t, ds) P(s, E) = \int_\Omega P(t, ds) P_1(s, E) = \int_\Omega P_1(t, ds) P_1(s, E) = P_1(t, E),$$

and
$$\underset{t\in\Omega,\, E\subset\Omega}{\text{l. u. b.}} \left| \frac{1}{n} \sum_{m=1}^{n} P^{(m)}(t, E) - P_1(t, E) \right| \leqq \frac{M}{n}, \quad n = 1, 2, \dots.$$

Moreover, if there exists no other proper value of T *of modulus* 1 *other than* 1, *then we have*

$$\underset{t\in\Omega,\, E\subset\Omega}{\text{l. u. b.}} \left| P^{(n)}(t, E) - P_1(t, E) \right| \leqq \frac{M}{(1+\varepsilon)^n}, \quad n = 1, 2, \dots,$$

where M *and* ε *are positive constants which are independent of* n.

Theorem 1. $P_1(t, E)$ *is expressible in the form:*

$$P_1(t, E) = \sum_{a=1}^{l} y_a(t) x_a(E), \tag{1}$$

1) M. Fréchet: Sur l'allure asymptotique de la suite des itérés d'un noyau de Fredholm, Quart. Journ. of Math., **5** (1934), 106–144.

M. Fréchet: Sur l'allure asymptotique des densités itérés dans le problèmé des probabilités en chaîne, Bull. Soc. Math. France, **62** (1934), 68–83.

2) J. L. Doob: Stochastic process with an integral valued parameter, Trans. Amer. Math. Soc., **44** (1938), 87–150.

3) W. Doeblin: Sur les propriétés asymptotiques de mouvements régis par certains types de chaînes simples, Bull. math. Soc. Roumaine des Sci. **39-2** (1937), 3–61.

4) K. Yosida: Abstract integral equations and the homogeneous stochastic process, Proc. **14** (1938), 287–291.

S. Kakutani: Iteration of linear operations in complex Banach spaces, Proc. **14** (1938), 295–300.

where $\{x_\alpha(E)\}$ $(\alpha=1, 2, \dots, l)$ *is a system of real-valued completely additive set functions from* $(\mathfrak{M})$ *such that*

$$(2) \qquad T(x_\alpha)=x_\alpha\,, \quad x_\alpha \geqq 0\,, \quad x_\alpha(\Omega)=1\,, \quad x_\alpha \wedge x_\beta=0 \quad (\alpha\neq\beta),$$ [1]

and any $x(E)\in(\mathfrak{M})$ *which satisfies*

$$(3) \qquad T(x)=x\,, \quad x\geqq 0\,, \quad x(\Omega)=1$$

can be uniquely expressed as a linear combination:

$$(4) \qquad x(E)=\sum_{\alpha=1}^{l} c_\alpha x_\alpha(E)\,, \quad c_\alpha\geqq 0\,, \quad \sum_{\alpha=1}^{l} c_\alpha=1\,.$$

Moreover, $\{y_\alpha(t)\}$ $(\alpha=1, 2, \dots, l)$ *is a system of real-valued bounded Borel measurable functions from* (M^*) *such that*

$$(5) \qquad \bar{T}(y_\alpha)=y_\alpha\,, \quad y_\alpha\geqq 0\,, \quad \sum_{\alpha=1}^{l} y_\alpha(t)\equiv 1\,,$$ [2]

$$(6) \qquad \int_\Omega x_\alpha(dt)y_\beta(t)=1 \text{ or } 0 \text{ according as } \alpha=\beta \text{ or } \alpha\neq\beta,$$

and any $y(t)\in(M^*)$ (*with* $y\geqq 0$) *which satisfies*

$$(7) \qquad \bar{T}(y)=y$$

can be uniquely expressed as a linear combination:

$$(8) \qquad y(t)=\sum_{\alpha=1}^{l} d_\alpha y_\alpha(t)$$

(*with* $d_\alpha\geqq 0$).

Let us consider the sets $\bar{E}_\alpha=\underset{t}{E}[y_\alpha(t)=1]$ $(\alpha=1, 2, \dots, l)$. By the third relation of (5), these are mutually disjoint Borel sets. If we further take suitably the Borel set E_α in each $\bar{E}_\alpha$, and if we put $\Delta=\Omega-\sum_{\alpha=1}^{l} E_\alpha$, then we have

Theorem 2.

$$(9) \qquad x_\alpha(\bar{E}_\beta)=x_\alpha(E_\beta)=1 \text{ or } 0 \text{ according as } \alpha=\beta \text{ or } \alpha\neq\beta,$$

$$(10) \qquad P(t, \bar{E}_\alpha)=1 \text{ if } t\in\bar{E}_\alpha\,,$$

$$(11) \qquad \underset{t\in\bar{E}_\alpha}{\text{l. u. b.}}\, |\, P^{(n)}(t, E_\alpha)-1\,| \leqq \frac{M}{(1+\varepsilon)^n}\,, \quad n=1, 2, \dots,$$

$$(12) \qquad P(t, E_\alpha)=1 \text{ if } t\in E_\alpha\,,$$

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1) For any $x(E)\in(\mathfrak{M})$, $x\geqq 0$ means that we have $x(E)\geqq 0$ for any Borel set $E\subset\Omega$; and, in case $x_1\geqq 0$ and $x_2\geqq 0$, $x_1\wedge x_2=0$ means that there exist two Borel sets E_1 and E_2 such that $x_1(E_1)=x_1(\Omega)$, $x_2(E_2)=x_2(\Omega)$ and $E_1E_2=0$.

2) For any $y(t)\in(M^*)$, $y\geqq 0$ means that we have $y(t)\geqq 0$ for any $t\in\Omega$.

$$\text{(13)} \qquad \underset{t \in \bar{E}_{a_1}, E \subset \Omega}{\text{l. u. b.}} \left| \frac{1}{n} \sum_{m=1}^{n} P^{(m)}(t, E) - x_a(E) \right| \leqq \frac{M}{n}, \quad n = 1, 2, \cdots,$$

$$\text{(14)} \qquad \begin{cases} t \in \bar{E}_a, \quad E \subset E_a \textit{ and } \operatorname{mes}(E) > 0 \textit{ imply the existence of} \\ \textit{a positive integer } n = n(t, E) \textit{ such that } P^{(n)}(t, E) > 0, \end{cases}$$

$$\text{(15)} \qquad \underset{t \in \Omega}{\text{l. u. b.}}\; P^{(n)}(t, \Delta) \leqq \frac{M}{(1+\varepsilon)^n}, \quad n = 1, 2, \cdots,$$

where M and ε are positive constants which are independent of n.

Remark. (10) means that each point $t \in \bar{E}_a$ is transferred inside $\bar{E}_a$, and (13) means that $\frac{1}{n} \sum_{m=1}^{n} P^{(m)}(t, E)$ converges uniformly to the limit which is independent of the initial point $t \in \bar{E}_a$. Because of these properties, $\bar{E}_a$ is called the *ergodic part* of Ω. Furthermore, (11) means that each point $t \in \bar{E}_a$ is transferred finally into E_a, and (12) means that each point $t \in E_a$ is transferred indside E_a. Moreover, by (14), E_a is indecomposable into two sets with the property (12). Hence E_a may be called the *ergodic kernel* of Ω. Lastly, by the property (15), Δ is called the *dissipative part* of Ω.

It is to be noted that E_a is determined only up to a set of measure zero, while $\bar{E}_a$ is strictly determined by the relation $\bar{E}_a = \underset{t}{E}[y_a(t) = 1]$. This has its reason in the fact that E_a is defined by a set function $x_a(E)$ from $(\mathfrak{M})$, while $\bar{E}_a$ is determined by a bounded measurable function $y_a(t)$ from (M^*).

Theorem 3. For each a, there exists a positive integer d_a such that $x_a(E)$ and $y_a(t)$ are decomposed into the form:

$$\text{(16)} \qquad x_a(E) = \frac{1}{d_a} \sum_{i=1}^{d_a} x^*_{a_i}(E),$$

$$\text{(17)} \qquad y_a(t) = \sum_{i=1}^{d_a} y^*_{a_i}(t),$$

where $\{x^*_{a_i}(E)\}$ $(i = 1, 2, \cdots, d_a)$ is a system of real-valued completely additive set functions from $(\mathfrak{M})$ which satisfy

$$\text{(18)} \qquad x^*_{a_i} \geqq 0, \quad x^*_{a_i}(\Omega) = 1, \quad x^*_{a_i} \wedge x^*_{a_j} = 0 \quad (i \neq j),$$

$$\text{(19)} \qquad T(x^*_{a_i}) = x^*_{a_{i+1}}, \quad i = 1, 2, \cdots, d_a \quad (a_{d_a+1} = a_1),$$

and $\{y^*_{a_i}(t)\}$ $(i = 1, 2, \cdots d_a)$ is a system of real-valued bounded Borel measurable functions from (M^*) which satisfy

$$\text{(20)} \qquad y^*_{a_i} \geqq 0,$$

$$\text{(21)} \qquad \bar{T}(y^*_{a_{i+1}}) = y^*_{a_i}, \quad i = 1, 2, \cdots, d_a \quad (a_{d_a+1} = a_1).$$

Let us again consider the sets $\bar{E}^*_{a_i} = \underset{t}{E}[y^*_{a_i}(t) = 1]$. These are also mutually disjoint Borel sets contained in $\bar{E}_a$. But the relation $\sum_{i=1}^{d_a}\bar{E}^*_{a_i} = \bar{E}_a$ is not always true; for, $\Delta_a \equiv \bar{E}_a - \sum_{i=1}^{d_a}\bar{E}^*_{a_i}$ is the set of all $t \in \bar{E}_a$ such that $y^*_{a_i}(t) < 1$ for $i=1, 2, \dots, d_a$, and this is not necessarily empty. If we further take suitably the Borel sets $E^*_{a_i}$ in each $\bar{E}^*_{a_i}$, then we have

Theorem 4.

(22) $x^*_{a_i}(\bar{E}^*_{a_j}) = x^*_{a_i}(E^*_{a_j}) = 1$ *or* 0 *according as* $i=j$ *or* $i \neq j$,

(23) $P(t, \bar{E}^*_{a_{i+1}}) = 1$ *if* $t \in \bar{E}^*_{a_i}$, $i=1, 2, \dots, d_a$ $(a_{d_{a+1}} = a_1)$,

(24) $\underset{t \in \bar{E}^*_{a_i}, E \subset \Omega}{\text{l. u. b.}} | P^{(nd_a)}(t, E) - x^*_{a_i}(E) | \leqq \frac{M}{(1+\varepsilon)^n}$, $n=1, 2, \dots$,

(25) $P(t, E^*_{a_{i+1}}) = 1$ *if* $t \in E^*_{a_i}$, $i=1, 2, \dots, d_a$ $(a_{d_{a+1}} = a_1)$,

(26) $$E_a = \sum_{i=1}^{d_a} E^*_{a_i}, \quad E^*_{a_i} = E_a \bar{E}^*_{a_i},$$

where M *and* ε *are positive constants which are independent of* n.

Remark. (23) means that each point $t \in \sum_{i=1}^{d_a}\bar{E}^*_{a_i}$ is transferred cyclically in $\bar{E}^*_{a_1}, \bar{E}^*_{a_2}, \dots, \bar{E}^*_{a_{d_a}}$. By (25), this is also the case for $E^*_{a_1}, E^*_{a_2}, \dots, E^*_{a_{d_a}}$. Moreover, by (24), $P^{(d_a)}(t, E)$ defines a Markoff process whose n-th iterate $P^{(nd_a)}(t, E)$ is uniformly convergent to a limit which is independent of the initial point $t \in \bar{E}^*_{a_i}$. We can also prove, by the same arguments, that we have

(27) $$\underset{t \in \bar{E}_a, E \subset \Omega}{\text{l. u. b.}} | P^{(nd_a)}(t, E) - \sum_{i=1}^{d_a} y^*_{a_i}(t) x^*_{a_i}(E) | \leqq \frac{M}{(1+\varepsilon)^n}, \quad n=1, 2, \dots,$$

where M and ε are positive constants which are independent of n.

In the proofs of all these theorems, the uniform ergodic theorem is indispensable; and in proving Theorem 1, some elementary considerations from the theory of semi-ordered Banach space are helpful. Moreover, in the proofs of Theorems 3, 4 and of the relations (11), (15) of Theorem 2, we have made an essential use of the fact that, under the condition (K), all the proper values of T of modulus 1 are roots of unity.[1]

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1) K. Yosida: Operator-theoretical treatment of the Markoff process II, Proc. **15** (1939), 127-136.