60. On One-parameter Groups of Operators.

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1. Theorem. Let E be a separable Banach space, and let $\{U_t\}$, $(-\infty < t < \infty)$ be a one-parameter group of operators on E to E such that: (1) $\| U_t \| = 1$, (2) $U_t U_s = U_{t+s}$ for any t, s, (3) $f(U_t x)$ is measurable in t for every $x \varepsilon E$ and for every $f \varepsilon \bar{E}$. Then there exist the operators R_z (resolvents) and A, which satisfy the following properties:

(1) R_z is defined for every complex number z, with $\mathcal{I}_m(z) \neq 0$,

(2) R_z is a bounded, linear operator on E to E, and $\| R_z \| \leqq \dfrac{1}{|\mathcal{I}_m(z)|}$,

(3) $(z-z') R_z R_{z'} = R_z - R_{z'}$, for every z, z' with $\mathcal{I}_m(z) \neq 0$, $\mathcal{I}_m(z') \neq 0$,

(4) $R_z x = 0$ implies $x = 0$, for any z;

(5) A is a closed linear oparator on E to E, whose domain $D(A)$ is dense in E, and

$$(A - zI) \cdot R_z = I, \qquad R_z(A - zI) = I \ \big(\text{in } D(A)\big),$$

(6) For any $x \varepsilon D(A)$, $\displaystyle \lim_{t \to 0} \frac{U_t - 1}{t} \cdot x = A \cdot x$.

We will prove these results, following the method of M. H. Stone.[1] Recently similar facts were obtained by I. Gelfand.[2] But the method is completely different from ours.

2. Proof: Let $\psi(\tau; z)$ be defined by

$$\psi(\tau; z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\lambda - z} e^{-i\lambda\tau} d\lambda \qquad \big(\mathcal{I}_m(z) \neq 0\big)$$

$$= \begin{cases} 0 & \tau > 0 \\ ie^{-iz\tau} & \tau < 0 \end{cases} \big(\mathcal{I}_m(z) > 0\big), \quad = \begin{cases} -ie^{-iz\tau} & \tau > 0 \\ 0 & \tau < 0 \end{cases} \big(\mathcal{I}_m(z) < 0\big).$$

Then

(i) $\displaystyle \frac{1}{\lambda - z} = \int_{-\infty}^{\infty} \psi(\tau; z) e^{i\lambda\tau} d\tau$,

(ii) $\displaystyle (z - z') \int_{-\infty}^{\infty} \psi(\tau; z) \psi(\sigma - \tau; z') d\tau = \psi(\sigma; z) - \psi(\sigma; z')$,

(iii) $\overline{\psi(\tau; z)} = \psi(-\tau; \bar{z})$.

We define $F(f)$ by

$$F(f) = \int_{-\infty}^{\infty} \psi(\tau; z) f(U_\tau x) d\tau, \quad f \varepsilon \bar{E}, \quad x \varepsilon E \text{ and } \mathcal{I}_m(z) \neq 0.$$

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1) M. H. Stone, Linear Transformations in Hilbert Space, 1932, Chap. IV, V; Annals of Math., **33** (1932), pp. 643–648.

J. von Neumann, Annals of Math., **33** (1932), pp. 567–573.

2) Gelfand, C. R. U. R. S. S., **25** (1939).

As a functional on $\bar{E}$, $F(f)$ is weakly continuous for every z $\left(\mathcal{I}_m(z) \neq 0\right)$. For, if $f_n \to f$ weakly in the sense that $f_n(x) \to f(x)$ for every $x \varepsilon E$, then $\|f_n\| \leqq M$ for some M (independent of n), and the functions of τ: $f_n(U_\tau x)$ are uniformly bounded. Consequently $\psi(\tau; z) = f_n(U_\tau x)$ are uniformly integrable in $(-\infty, \infty)$. As $f_n(U_\tau x) \to f(U_\tau x)$ for all τ, it follows

$$F(f_n) = \int_{-\infty}^{\infty} \psi(\tau; z) f_n(U_\tau x)\, d\tau \to F(f) = \int_{-\infty}^{\infty} \psi(\tau; z) f(U_\tau x)\, d\tau .$$

Therefore, by a theorem of Banach, there exists an $x_z \varepsilon E$ such that $F(f) = f(x_z)$, for any $f \varepsilon \bar{E}$. We define R_z by $x_z = R_z x$.

Evidently R_z is defined and additive over the whole E, and

$$|f(R_z \cdot x)| = \left| \int_{-\infty}^{\infty} \psi(\tau; z) f(U_\tau x) d\tau \right| \leqq \frac{\|f\| \cdot \|x\|}{|\mathcal{I}_m(z)|} ,$$

and

$$\|R_z\| \leqq \frac{1}{|\mathcal{I}_m(z)|} .$$

R_z is linear. For z, z' with $\mathcal{I}_m(z) \neq 0$, $\mathcal{I}_m(z') \neq 0$,

$$\begin{aligned} f(R_z R_{z'} x) &= \int_{-\infty}^{\infty} \psi(t; z') f(U_t R_z x)\, dt \\ &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \psi(s; z) \psi(t; z') f(U_{t+s} x)\, ds \\ &= \int_{-\infty}^{\infty} f(U_t x)\, dt \int_{-\infty}^{\infty} \psi(s; z) \psi(t-s; z')\, ds , \end{aligned}$$

and by (ii), **2**,

$$(z-z') f(R_z R_{z'} x) = f(R_z x) - f(R_{z'} x) .$$

If $R_{z_1} = 0$, for certain z_1 with $\mathcal{I}_m(z_1) \neq 0$, then by virtue of (2), $R_z x = 0$ for all $z\left(\mathcal{I}_m(z) \neq 0\right)$, and

$$0 = f(R_z x) - f(R_{\bar{z}} x) = \int_{-\infty}^{\infty} f(U_t x) e^{-\eta|t|} e^{-i\xi t} dt , \quad z = \xi + i\eta , \quad \eta \neq 0 .$$

As $f(U_t x)$ is continuous in t,

$$f(U_t x) = 0 , \quad \text{for all } t ; \text{ in particular, } f(U_0 x) = f(x) = 0 ,$$

and, as this holds for all $f \varepsilon \bar{E}$, we must have $x = 0$.

3. Proof (continued). The operator A.

i) Definition. Let z_0 be a complex number, with $\mathcal{I}_m(z_0) \neq 0$, and let A take $y = R_{z_0} x$ into $x + z_0 R_{z_0} \cdot x$, where x is any element of E:

$$y = R_{z_0} x , \qquad Ay = x + z_0 R_{z_0} x .$$

Then the domain $D(A)$ of A is everywhere dense in E. For, if this is not the case, then there exists an $f \varepsilon \bar{E}$ such that $f \neq 0$ and $f(x) = 0$ for any $x \varepsilon D(A)$. Thus we have $f(R_{z_0} x) = 0$ for all $x \varepsilon E$, and consequently, for $\mathcal{I}_m(z_0) > 0$, and for any $x \varepsilon E$,

$$0=f(R_{z_0}U_tx)=i\int_{-\infty}^{0}e^{iz_0\tau}f(U_\tau U_tx)\,d\tau=ie^{-iz_0t}\int_{-\infty}^{t}e^{iz_0s}f(U_sx)\,ds\,,$$

for every t, and we have, for all t, $f(U_tx)=0$.

As x is any element of E, we have $f=0$.

ii) $$R_z(A-zI)y=(A-zI)\ R_zy=y\,,\quad (y=R_{z_0}x)\,.$$

In fact, for $y=R_{z_0}x$,

$$R_z(A-zI)\ y=R_z(x+z_0R_{z_0}x-zR_{z_0}x)$$

and
$$(A-zI)\cdot R_zy=A\big(R_{z_0}(R_zx)\big)-(zR_zR_{z_0}x)$$
$$=R_zx+z_0R_{z_0}R_zx-z\cdot R_zR_{z_0}x\,.$$

Consequently,

$$R_z(A-zI)\cdot y=(A-zI)\cdot R_zy=R_zx+(z_0-z)\ R_zR_{z_0}x$$
$$=R_zx+(R_{z_0}x-R_zx)=R_{z_0}x=y\,.$$

iii) A is additive (evident).

iv) A is closed. Let $y_n=R_{z_0}x_n\to y$, $Ay_n=x_n+z_0R_{z_0}x_n\to\bar{y}$. Then $x_n=Ay_n-z_0y_n\to\bar{y}-z_0y$ $(=x)$, and $Ay_n=x_n+z_0R_{z_0}x_n\to x+z_0R_{z_0}x$. Consequently, $A\cdot R_{z_0}x=x+z_0R_{z_0}x$; this proves the closedness.

v) The domain of A is independent of each choice of z_0:
For, if $z_0\to A$, $z_0'\to A'$ ($\to$ is the sense of the definition), then, at first $D(A')\subset D(A)$. In fact, for any $y'=R_{z_0'}x$, take $y=R_{z_0}x\,\varepsilon\,D(A)$ Then, $y-y'R_{z_0}x-R_{z_0'}x=(z-z_0')\ R_{z_0}R_{z_0'}x$. As y, $R_{z_0}y'\,\varepsilon\,D(A)$, and $D(A)$ is linear, $y'=y-(z_0-z_0')R_{z_0}y'\,\varepsilon\,D(A)$. By symmetry, we have $D(A')=D(A)$.

On the other hand, we have

$$R_z(A-zI)=I\,,\quad R_z(A'-zI)=I\quad\big(\text{in }D(A)\equiv D(A')\big)$$

and
$$R_z(A-A')=0\,,\quad\text{therefore}\quad A\equiv A'\quad\big(\text{cf. (4)}\big)\,.$$

vi) $$\lim_{t\to0}\frac{U_t-I}{t}y=A\cdot y\,,\quad\text{for}\quad y\,\varepsilon\,D(A)\,.$$

Let
$$y=R_{z_0}\cdot x\,,\quad\text{and}\quad \mathcal{I}_m(z_0)>0\,,\quad\text{say.}$$

$$f\Big(\frac{U_t-I}{t}R_{z_0}\cdot x\Big)=\int_{-\infty}^{\infty}f(U_\tau x)\frac{\psi(\tau-t;\,z_0)-\psi(\tau;\,z_0)}{t}\,d\tau$$
$$=\int_{-\infty}^{0}+\int_{0}^{\varepsilon}+\int_{\varepsilon}^{\infty},$$

$$\int_0^{\varepsilon}f(U_\tau x)\frac{\psi(\tau-t;\,z_0)-\psi(\tau;\,z_0)}{t}\,d\tau$$
$$=\int_0^{\varepsilon}f(U_\tau x)\frac{\psi(\tau-t;\,z_0)}{t}\,d\tau$$
$$=\frac{1}{t}\int_{-t}^{0}(U_{t+\sigma}x)\,\psi(\sigma;\,z_0)\,d\sigma\,;$$

$$\lim_{t\to 0}\frac{1}{t}\int_{-t}^{0}f(U_{t+\sigma}x)\,\psi(\sigma;z_0)\,d\sigma$$

$$=\lim_{t\to 0}\frac{1}{t}\int_{-t}^{0}f(U_\sigma x)\,\psi(\sigma;z_0)\,d\sigma$$

$$=f(U_0x)\cdot\psi(0;z_0)=f(x)\,.$$

We have

$$\lim_{t\to 0}f\left(\frac{U_t-I}{t}R_{z_0}\cdot x\right)=f(x)+z_0f(R_{z_0}x)=f(Ay)\,,$$

by definition.