PAPERS COMMUNICATED

58. *On the Division of a Probability Law.*

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1. If a random variable X is represented as a sum X_1+X_2 of independent variables X_1 and X_2, in other words, if the characteristic function of X

$$f(t)=\int_{-\infty}^{\infty}e^{itx}d\sigma(x)$$

$\sigma(x)$ being the distribution function of X, is represented as a product $f_1(t)f_2(t)$ of characteristic functions $f_1(t)$ and $f_2(t)$ of X_1 and X_2 respectively, X is said to be divisible by X_1 or X_2. The division of X by some random variable is not necessarily determined uniquely what was proved by Gnedenko and Khintchine.[1] That is, there exist characteristic functions $f(t)$, $f_1(t)$, $f_2(t)$ and $f_3(t)$ such that

$$f(t)=f_1(t)f_2(t)=f_1(t)f_3(t)\,, \tag{1}$$

where $f_2(t)$ is not identically equal to $f_3(t)$. But it was shown by P. Lévy that if X is indefinitely divisible, then the division is uniquely determined. The purpose of this paper is to discuss the unicity of divisibility in terms of a distribution function of X.

2. If there exists a t_0 such that $f_2(t_0)\neq f_3(t_0)$, then since a characteristic function is continuous there exists an interval $a<t<b$ in which $f_2(t)\neq f_3(t)$. Since if (1) holds then $f_1(t)\{f_2(t)-f_3(t)\}=0$, in this case $f_1(t)$ or $f(t)$ vanishes in $a<t<b$. Thus the sufficient conditions for the non-vanishing of $f(t)$ in any interval is also the sufficient conditions for the unique determination of division of X. Hence known results[2] on non-vanishing of function yield the following theorems.

Theorem 1. Let $\sigma(x)$ be the distribution function of a random variable X and let $\theta(u)$ be a positive, non-decreasing function defined in $(0, \infty)$ *such that*

$$\int_1^{\infty}\frac{\theta(u)}{u^2}du=\infty\,. \tag{2}$$

If for some constant a (>0)

$$\sigma(-u+a)-\sigma(-u-a)=O\big(\exp(-\theta(u))\big) \tag{3}$$

and X is divisible by some variable, then the quotient is unique.

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1) B. Gnedenko, Sur les fonctions caractéristiques, Bull. l'Univ. Moscou, **1** (1937), P. Lévy, Théorie de l'addition des variables aléatoires, (1937), pp. 189–190.

2) T. Kawata, Non-vanishing of functions and related problems. Tohoku Math. Journ., **46** (1940), Theorems 4 and 11.

Theorem 2. Let $\sigma(x)$ *be the distribution function of a random variable* X *and be a step function in* $(-\infty, 0)$ *with point spectra* a_n $(a<0)$ *and be unconditioned on the behaviour over* $(0, \infty)$. *If*

$$\lim_{n\to\infty} |a_n - a_{n+1}| = \infty , \tag{4}$$

and X *is divisible by some variable, then the quotient is unique.*

3. First it will be proved that *even when* $\sigma(x)$ *has only point spectra on* $(-\infty, \infty)$, *the conclusion of Theorem 2 does not necessarily hold if* (4) *is not assumed.*

Let $f_1(t)$ and $f_2(t)$ be continuous even periodic functions with period 2π such that

$$f_1(t) = \begin{cases} -\dfrac{2}{\pi}t+1 , & 0 \leqq t \leqq \dfrac{\pi}{2} , \\ 0 , & \dfrac{\pi}{2} \leqq t \leqq \pi , \end{cases}$$

$$f_2(t) = \begin{cases} -\dfrac{2}{\pi}t+1 , & 0 \leqq t \leqq \dfrac{\pi}{2} , \\ \dfrac{2}{\pi}t-1 , & \dfrac{\pi}{2} \leqq t \leqq \pi , \end{cases}$$

and let the Fourier coefficients of $f_1(t)$ and $f_2(t)$ be c_n and d_n respectively. Then easy calculation shows

$$c_n = \frac{2}{n^2\pi^2}\left(1-\cos\frac{n\pi}{2}\right),$$

$$d_n = \frac{2}{n^2\pi^2}(1-\cos n\pi) .$$

Thus c_n and d_n are non-negative and $\sum c_n = \sum d_n = 1$. Hence $f_i(t)$ are the characteristic functions of random variables X_i $(i=1, 2)$ and the spectra of f_i are integers. Further by known results, X_1+X_1 also has a distribution having integral point spectra only. From the definitions of f_1 and f_2 it is obvious that

$$f_1^2 = f_1 . f_2$$

and f_1 is not identically equal to f_2. Thus

$$X = X_1 + X_2 = X_1 + X_2$$

which proves our assertion.

4. Here we shall prove the condition (2) in Theorem 1 is best possible of its kind.

Theorem 3. Let $\sigma(u)$ *be any positive, non-decreasing function defined in* $(0, \infty)$ *such that*

$$\int_1^\infty \frac{\theta(u)}{u^2} du < \infty . \tag{5}$$

Then for every positive number a, *there exists a random variable* X *such that*

$$X = X_1 + X_2 = X_1 + X_3 ,$$

where X_1, X_2 and X_3 are certain variables and $X_2 \neq X_3$ and further the distribution $\sigma(x)$ of X satisfies

$$(6) \qquad \sigma(-u+a)-\sigma(-u-a)=O\Big(\exp\big(-\theta(u)\big)\Big) \quad \text{for} \quad u\to\infty\,.$$

For proof we require following theorems which will be stated as lemmas.

Lemma 1. Let $\theta(u)$ be the function in Theorem 3 and l be any positive number. Then there exists a non-null function $G(x)$ such that

$$G(x)=O\Big(\exp\big(-\theta(|x|)\big)\Big) \quad \text{for} \quad x\to\pm\infty\,,$$

and the ordinary Fourier transform of $G(x)$

$$F(u)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}G(x)e^{-iux}dx$$

vanishes for $|u|>l$.

This was obtained by A. Ingham[1] and N. Levinson.[2]

Lemma 2. If $\psi(x)\in L_2(-\infty,\infty)$ and be non-null, then

$$\varphi(t)=\frac{1}{A}\int_{-\infty}^{\infty}\psi(x)\overline{\psi(x+t)}\,dx\,, \qquad \left(A=\int_{-\infty}^{\infty}|\psi(x)|^2dx\right)$$

is the characteristic function of a random variable.

This is a particular case of a more complete theorem concerning the characterization of the characteristic function of a random variable which is due to A. Khintchine.[3]

Before we proceed to prove Theorem 3, we shall prove the following theorem using Lemma 1 and 2.

Theorem 4. If $\theta(u)$ is the function in Theorem 3 and l is any positive number, then there exists a distribution $\sigma(u)$ such that it satisfies the condition (6) for every a (>0) and its characteristic function $\Lambda(t)$ vanishes in $|t|>l$.

We consider $\theta(2u)$ which obviously satisfies the conditions imposed on $\theta(u)$ and we consider $F(u)$ in Lemma 1 replacing l by $l/2$ and $\theta(u)$ by $\theta(2u)$. $F(u)$ vanishes for $|u|>l/2$. Put

$$\Lambda(t)=\frac{1}{A}\int_{-\infty}^{\infty}F(x)\overline{F(x+t)}\,dx\,, \qquad A=\int_{-\infty}^{\infty}|F(x)|^2dx\,.$$

Then $\Lambda(t)$ is a characteristic function of a random variable by Lemma 2 and as easily verified, $\Lambda(t)=0$ for $|t|>l$.

Now direct computation shows

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1) A. E. Ingham, A note on Fourier transform, Journ. London Math. Soc., **9** (1936).

2) N. Levinson, On a class of non-vanishing functions, Proc. London Math. Soc., **41** (1936).

N. Levinson, A theorem relating non-vanishing and analytic functions, Journal of Math. and Physics **16** (1938).

3) A. Khintchine, Zur Kennzeichnung der charakteristischen Funktionen, Bull. l'Univ. Moscou, **1** (1937).

$$\sigma(-u+a)-\sigma(-u-a)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Lambda(x)\frac{\sin ax}{ax}e^{iux}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-l}^{l}\frac{\sin ax}{ax}e^{iux}dx\int_{-l/2}^{l/2}\overline{F(x+t)}F(t)\,dt$$

$$=\int_{-l/2}^{l/2}F(t)\,dt\int_{-l}^{l}\overline{F(x+t)}\frac{1}{\sqrt{2\pi}}\frac{\sin ax}{ax}e^{iux}dx,$$

which becomes by Parseval theorem in the theory of Fourier transform

$$\frac{1}{2a}\int_{-l/2}^{l/2}F(t)\,dt\int_{-a}^{a}G(u-y)\,e^{i(u-y)t}dy,$$

$G(t)$ being the one in Lemma 1,

$$=O\Big(\int_{-l/2}^{l/2}|F(t)|\,dt\int_{-a}^{a}|G(u-y)|\,dy\Big)$$

$$=O\Big(\int_{-a}^{a}\exp\big(-\theta(2u-2y)\big)\,dy\Big)=O\Big(\exp\big(-\theta(2u-2a)\big)\Big)$$

$$=O\Big(\exp\big(-\theta(u)\big)\Big),\quad \text{for}\quad u>2a.$$

Thus the theorem is proved.

5. We now proceed to prove Theorem 3. We consider $\theta(2u)$ instead of $\theta(u)$. By Theorem 4, taking $l=\pi/2$, we can find a function $F(x)$ such that

(6) $$F(u)=0 \quad \text{for} \quad |u|>\pi/2$$

and $$\Lambda(t)=\frac{1}{A}\int_{-\infty}^{\infty}F(x)\overline{F(x+t)}\,dx,\qquad A=\int_{-\infty}^{\infty}|F(x)|^2dx$$

satisfies

(7) $$\Lambda(t)=0 \quad \text{for} \quad |t|>\pi,$$

and is a characteristic function of a distribution $\sigma(x)$ which satisfies that for every $a>0$

(8) $$\sigma(-u+a)-\sigma(-u-a)=O\Big(\exp\big((-\theta(2u)\big)\Big),$$

where $\theta(u)$ is the one in Theorem 3.

Now by (6) we can write

(9) $$\Lambda(t)=\frac{1}{A}\int_{-\pi/2}^{\pi/2}F(x)\overline{F(x+t)}\,dx.$$

Let $M(t)$ be a periodic function with period 2π which coincides with $\Lambda(t)$ in $|t|\leqq\pi$. Since $\Lambda(\pi)=\Lambda(-\pi)=0$, $M(t)$ is continuous. If c_n denotes the Fourier coefficient of $M(t)$, then

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}M(t)\,e^{-int}dt=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-int}dt\int_{-\pi/2}^{\pi/2}F(x)\overline{F(x+t)}\,dx$$

$$=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}F(x)\,e^{inx}dx\int_{-\pi+x}^{\pi+x}\overline{F(t)}\,e^{-int}dt.$$

Now since $\pi+x \geqq \frac{\pi}{2}$, $-\pi+x \leqq \frac{\pi}{2}$ and $F(t)=0$ for $|t|>\pi/2$, we have

$$c_n = \int_{-\pi/2}^{\pi/2} F(x)\, e^{inx}\, dx \int_{-\pi/2}^{\pi/2} \overline{F(t)}\, e^{-int}\, dt$$
$$= \frac{1}{2\pi} \left| \int_{-\pi/2}^{\pi/2} F(x)\, e^{inx}\, dx \right|^2 \geqq 0\,.$$

Thus $c_n \geqq 0$ and $\sum_{n=-\infty}^{\infty} c_n = M(0) = 1$. Hence

$$M(t) = \sum_{n=-\infty}^{\infty} c_n e^{-int}\,,$$

is the characteristic function of a distribution such that it has only point spectra contained in the sequence of all integers and the jump at a spectrum n is c_n. Obviously $M(t)$ is not identically equal to $\Lambda(t)$ and

$$\Lambda^2(t) = M(t)\, \Lambda(t)\,. \tag{10}$$

Hence if X_1 and X_2 are independent random variables whose distributions are $\Lambda(t)$ and $M(t)$ respectively, then manifestly we have

$$X = X_1 + X_1 = X_1 + X_2\,, \tag{11}$$

where X_1 and X_2 are not identically equal. Let the distribution of X be $\tau(u)$. We have

$$\tau(u) = \int_{-\infty}^{\infty} \sigma(u-t)\, d\sigma_2(t)\,,$$

where $\sigma_2(t)$ is the distribution function of X_2.

We have then

$$\tau(-u+a) - \tau(-u-a) = \sum_{n=-\infty}^{\infty} c_n \Big(\sigma(-u-n+a) - \sigma(-u-n-a) \Big)$$
$$= \sum_{n=-\infty}^{-[u/2+1]} + \sum_{n=-[u/2+1]+1}^{\infty}$$
$$= S_1 + S_2\,,$$

say. Since

$$\sum_{-\infty}^{\infty} \Big(\sigma(-u-n+a) - \sigma(-u-n-a) \Big) \leqq [a]+1\,,$$

and

$$c_{-n} = O\Big(\exp\big(-2\theta(2|n|)\big) \Big)$$

which is easily seen from the construction of $F(x)$ in Lemma 1 (noticing that we are considering $\theta(2u)$ instead of $\theta(u)$), we get

$$|S_1| = \Big| \sum_{n=-\infty}^{-[u/2+1]} \Big| \leqq O\Big(\exp\big(-2\theta(2[u/2+1])\big) \Big)$$
$$\sum_{n=-\infty}^{\infty} \Big(\sigma(-u-n+a) - \sigma(-u-n-a) \Big)$$
$$= O\Big(\exp\big(-2\theta(u)\big) \Big) = O\Big(\exp\big(-\theta(u)\big) \Big)\,.$$

$$|S_2| \leqq \sum_{-[u/2+1]+1}^{\infty} c_n\left(\sigma(-u-n+a)-\sigma(-u-n-a)\right)$$

$$\leqq \sum_{-[u/2+1]+1}^{\infty} c_n O\left(\exp\left(-\theta(u)\right)\right) = O\left(\exp\left(-\theta(u)\right)\right)\sum_{n=-\infty}^{\infty} c_n$$

$$= O\left(\exp\left(-\theta(u)\right)\right).$$

Hence we finally get

$$\tau(-u+a)-\tau(-u-a) = O\left(\exp\left(-\theta(u)\right)\right).$$

Thus the theorem is completely proved.