78. A Converse of Lebesgue's Density Theorem.

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I. The main object of the present note is to see that a converse of Lebesgue's density theorem holds.

We shall consider, for brevity, sets of points in a Euclidean plane R^2 only. But the results which will be obtained can obviously be extended to spaces R^m of any number of dimensions.

The Lebesgue outer measure of a set E in R^2 will be denoted by $|E|$. Let x be a point of R^2 and Q an arbitrary closed square containing x with sides parallel to the coordinate-axes.

We shall denote by $\bar{D}(x, E)$ and $\underline{D}(x, E)$ the superior and the inferior limit respectively of the ratio $|QE|/|Q|$ as the diameter of Q tends to 0 or $|Q|\to 0$, and shall call them the upper and the *lower density* of E at x respectively. If they are equal to each other at x, then the common value will be called the density of E at x. The points at which the density of E are equal to 0 are termed points of dispersion for E.

It is well known by Lebesgue's density theorem that, *if a set of points are measurable, almost every point of its complementary set is a point of dispersion for the given set.*[1]

II. We shall prove the following theorem which evidently contains a converse of the above proposition.

Theorem 1. *Let E be a point-set whose lower density is* 0 *at almost every point of the complementary set of E. Then the set E is measurable.*

Proof. We can obviously assume, without loss of generality, that the set E is bounded. Let G be a bounded open set containing E and ε a given positive number. From the assumption of the present theorem, there exists, for almost every point x of $H=G-E$, a sequence of squares $\{Q_n(x)\}$ such that

$$(1)\qquad \frac{|E\cdot Q_n(x)|}{|Q_n(x)|}<\varepsilon,\ x\in Q_n(x),\ |Q_n(x)|\to 0\ (n\to\infty)\ \text{and}\ Q_n(x)\subset G.$$

Denoting by $\mathfrak{F}$ the family of all the squares which belong to any one of such sequences, we find that $\mathfrak{F}$ covers the set H almost everywhere in the sense of Vitali.[2] According to the covering theorem of Vitali[2] we can extract from $\mathfrak{F}$ a finite or enumerable sequence $\{Q_n\}$ of squares no two of which have common points, such that

$$|H-\sum_n Q_n|=0\,.$$

1) For example, S. Saks. Théorie de l'intégrale (1933), p. 55.
2) Saks. Loc. cit. 33–35.

$\sum_n Q_n$ being measurable, we find, from the relation above

$$(2) \qquad |H| = |H - \sum Q_n| + |H \sum Q_n| = |H \sum Q_n| \\ \leqq |\sum Q_n| .$$

On the other hand, as $G - \sum Q_n \supset E(G - \sum Q_n) = E - \sum Q_n$, we have

$$|G - \sum Q_n| \geqq |E - \sum Q_n| ,$$

and since $|E - \sum Q_n| = |E| - |E \sum Q_n|$ on account of the measurability and the boundedness of $\sum Q_n$, we obtain

$$|G - \sum Q_n| \geqq |E| - |E \sum Q_n| = |E| - \sum |Q_n E| .$$

From the first inequality of (1) and the relation above, it follows that

$$(3) \qquad |G - \sum Q_n| > |E| - \sum \varepsilon \cdot |Q_n| \geqq |E| - \varepsilon \cdot |G| .$$

By (3) and (2), we have

$$|G| = |G - \sum Q_n| + |\sum Q_n| \\ \geqq |E| - \varepsilon |G| + |H| .$$

In the above inequality, making ε tend to 0, we obtain

$$|G| \geqq |E| + |H| ,$$

and so we have

$$(4) \qquad |G| = |E| + |G - E| ,$$

since $|G| \leqq |E| + |H|$ is obvious, by which we ascertain the measurability of E, for, let η be an arbitrarily given positive number and G an open set such that

$$(5) \qquad |G| < |E| + \eta \quad \text{and} \quad G \supset E ,$$

and we will find, from (4) and (5),

$$|G - E| = |G| - |E| < \eta .$$

In our theorem, we assume only $\underline{D}(x, E) = 0$ and do not assume $D(x, E) = 0$. But as the result of our theorem, E is found to be measurable and accordingly, on account of the theorem of Lebesgue on density, the latter relation holds almost everywhere outside of E.

As the corollary of Theorem 1, we obtain the following:

Theorem 2. *Let E be a set of points. Suppose that there exists a measurable set A containing E and satisfying the condition that at almost every point of $A - E$ the lower density of E is* 0.

Then E is measurable.

Proof. Let A^c and E^c be complementary sets of A and E respectively.

Then $$E^c = A^c + (A - E) .$$

A being measurable, from Lebesgue's density theorem, we have

$D(x, A)=0$ at almost every point x of A^c, from which we find that, since $A \supset E$, $D(x, E)=0$ holds also at almost every point of x of A^c.

On the other hand, by the assumption of the present theorem $\underline{D}(x, E)=0$ holds at almost every point x of $A-E$. Hence the set E satisfies the assumption of Theorem 1, and this completes the proof.

III. Let $f(x)$ be a finite function defined on a measurable set A. Given an arbitrary property (P) of a point x, let us denote, as usual, by $\underset{x}{E}[(P)]$ the set of all the points x of A that have this property. Suppose that, for each $\varepsilon>0$, we have, at almost every point y of A,

$$\underline{D}\left(y, \underset{x}{E}[f(x)-f(y) \geqq \varepsilon]\right)=0\,. \tag{6}$$

It is easy to see that this condition defines a wider class of functions than that of upper semi-continuous functions.

We shall show that $f(x)$ is measurable.

For this purpose, let us consider the set of $\underset{x}{E}[f(x) \geqq k]$ where k is an arbitrary constant. If $y \in A-\underset{x}{E}[f(x) \geqq k]$, then $f(y)<k$ or, putting $k-f(y)=\varepsilon$, $\varepsilon>0$.

By the obvious relation

$$\underset{x}{E}[f(x) \geqq k]=\underset{x}{E}[f(x) \geqq f(y)+\varepsilon]=\underset{x}{E}[f(x)-f(y) \geqq \varepsilon]$$

and (6), it follows that

$$\underline{D}\left(y, \underset{x}{E}[f(x) \geqq k]\right)=0$$

holds at almost every point y of $A-\underset{x}{E}[f(x) \geqq k]$.

This shows, in virtue of Theorem 2, the measurability of $\underset{x}{E}[f(x) \geqq k]$ and accordingly of $f(x)$.

Conversely, if a finite function $f(x)$ is measurable on a measurable set A, then by Denjoy's theorem,[1] we know that $f(x)$ is approximately continuous at almost every point of A, which means that at almost every point y of A, there exists a measurable subset of A whose density at y is 1 and by the points x of which the condition $\lim\limits_{x \to y} f(x)=f(y)$ is fulfilled.

Then, it is quite easy to see that (6) is fulfilled by $f(x)$ at almost every point y of A. Thus we have proved the following:

Theorem 3. *It is necessary and sufficient for a finite function $f(x)$ defined on a measurable set A to be measurable is that, for each $\varepsilon>0$, $f(x)$ fulfils the condition* (6) *at almost every point y of A.*

Recently, I. J. Good has proved a theorem[2] according to which: if $f(x)$ is a finite measurable function defined on a linear set A, then at almost every point y of A,

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1) Saks. Loc. cit. p. 232. There the density is considered in the strong sense which needs no change in our proof.

2) I. J. Good. The approximate local monotony of measurable functions, Proc. of the Cambridge Philosophical Soc., Vol. 36 (1940), 9-13, see esp. p. 10, section 5.

$$\underline{D}\left(y, \underset{x}{E}[f(x) > f(y)]\right) = 0. \tag{7}$$

While Good restricted himself to the case that A is a linear set, his result is a little more precise than that given above.

By the method already used, we shall show that a converse of his theorem holds even if the set A is not linear.

Theorem 4. *If a finite function $f(x)$ defined on a measurable set A satisfies* (7) *at almost every point y of A, then $f(x)$ is measurable.*

Proof. As in the proof of Theorem 3, we consider the set $\underset{x}{E}[f(x) \geqq k]$, where k is an arbitrary constant. Let y be any point of $A - \underset{x}{E}[f(x) \geqq k]$, then $f(y) < k$ and accordingly $\underset{x}{E}[f(x) \geqq k] \subset \underset{x}{E}[f(x) > f(y)]$.

Hence we have $\underline{D}\left(y, \underset{x}{E}[f(x) \geqq k]\right) \leqq \underline{D}\left(y, \underset{x}{E}[f(x) > f(y)\right)$, in which the left hand side is non negative while the right hand side is 0 at almost every point y of A. Therefore we obtain

$$\underline{D}\left(y, \underset{x}{E}[f(x) \geqq k]\right) = 0$$

for almost every point y of $A - \underset{x}{E}[f(x) \geqq k]$, which completes the proof on account of Theorem 2.